

# Short Course on Wind Energy – Scale Effects in Wind Turbines –

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# Scale Effects

# Scale Effects

## Applications:

- Understanding scaling effects in small–large wind turbines
- Scaling laws for wind tunnel models

## Tools:

- Buckingham  $\pi$ -Theorem (dimensional analysis)



# Dimensional Analysis: Buckingham $\pi$ -Theorem

## Buckingham $\pi$ -Theorem:

*Given the governing equation for a physical system defined by  $n$  physical variables, which are expressible in terms of  $k$  independent fundamental quantities, one can construct an equivalent equation involving a set of  $m = n - k$  dimensionless variables constructed from the original variables*

The dimensionless parameters provide relations which define the scaling laws



# Dimensional Analysis: Buckingham $\pi$ -Theorem

Consider the generic model:

$$f(\underbrace{p_1, \dots, p_n}_{\text{Physical parameters}}, \underbrace{a_1, \dots, a_k}_{\text{Fundamental quantities}}) = 0$$

The  $i$ -th **physical parameter** can be expressed in terms of  $k$  **independent fundamental quantities**:

$$p_i = a_1^{d_{1i}} \cdot \dots \cdot a_k^{d_{ki}}$$

$d_{i1}, \dots, d_{ik}$  = dimensions of  $p_i$  wrt fundamental quantities

The **Dimensional Matrix**  $\mathcal{D} \in \mathbb{R}^k \times \mathbb{R}^n$  contains as elements  $d_{ri}$  the  $r$ -th dimension of the  $i$ -th parameter



# Dimensional Analysis: Buckingham $\pi$ -Theorem

**Equivalent non-dimensional model:**

$$\phi(\pi_1, \dots, \pi_m) = 0, \quad m = n - k$$

j-th non-dimensional parameter:

$$\pi_j = p_1^{m_{1j}} \cdot \dots \cdot p_n^{m_{nj}}$$

The exponents are the components of matrix  $\mathcal{M} \in \mathbb{R}^n \times \mathbb{R}^m$  found from the solution of the following system of equations:

$$\mathcal{D} \mathcal{M} = \mathbf{0} \quad \text{i.e.:} \quad \mathcal{M} = \text{null}(\mathcal{D})$$

**Remarks:**

- Set of  $m=n-k$  parameters is not unique
- Selection of proper/representative non-dimensional parameters needs to be guided by physical considerations based on the model under consideration



# Similarity

Consider two systems:

$\mathbb{P}$  = **Physical system** (full scale)

$\mathbb{M}$  = **Model** (reduced scale)

with governing equations:

$$f(p_{1\mathbb{P}}, \dots, p_{n\mathbb{P}}, a_{1\mathbb{P}}, \dots, a_{k\mathbb{P}}) = 0$$

$$f(p_{1\mathbb{M}}, \dots, p_{n\mathbb{M}}, a_{1\mathbb{M}}, \dots, a_{k\mathbb{M}}) = 0$$

and equivalent non-dimensional relations obtained through Buckingham  $\pi$ -theorem:

$$\phi(\pi_{1\mathbb{P}}, \dots, \pi_{m\mathbb{P}}) = 0$$

$$\phi(\pi_{1\mathbb{M}}, \dots, \pi_{m\mathbb{M}}) = 0$$



# Similarity

$\mathbb{P}$  is **similar** to  $\mathbb{M}$  if:

$$\pi_{j\mathbb{P}} = \pi_{j\mathbb{M}}, \text{ for } j = 1, \dots, m$$

which provides a set of scaling relations:

$$p_{1\mathbb{P}}^{m_{1j}} \cdot \dots \cdot p_{n\mathbb{P}}^{m_{nj}} = p_{1\mathbb{M}}^{m_{1j}} \cdot \dots \cdot p_{n\mathbb{M}}^{m_{nj}}$$

Generic **scaling relation** for a model parameter:

$$p_{1\mathbb{M}} = p_{1\mathbb{P}} \left( \frac{p_{2\mathbb{P}}}{p_{2\mathbb{M}}} \right)^{\frac{m_{2j}}{m_{1j}}} \dots \left( \frac{p_{n\mathbb{P}}}{p_{n\mathbb{M}}} \right)^{\frac{m_{nj}}{m_{1j}}}$$



# Similarity and Scaling

Typically, scaling is based on the assignment of a **scaling parameter**:

$$n = \frac{l_M}{l_P} \begin{array}{l} \longrightarrow \text{Characteristic length of the model} \\ \longrightarrow \text{Characteristic length of full scale system} \end{array}$$

Accordingly, the general scaling relation can be expressed as:

$$p_{1M} = p_{1P} n^{\alpha_{12}} \cdot \dots \cdot n^{\alpha_{1n}} \quad p_{1M} = p_{1P} n^{\alpha_1}$$

The scaling process reduces to a **linear transformation of parameters**:

$$\mathbf{p}_M = \mathcal{S} \mathbf{p}_P$$

**Scaling matrix:**  $\mathcal{S} = \text{diag}(n^{\alpha_1}, \dots, n^{\alpha_n})$





# Wind Turbine Dimensional Analysis

Derive equations of motion, apply Buckingham  $\pi$ -theorem using as fundamental quantities

**Mass, Length, Time**

Non-dimensional equations of motion:

$$\phi(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \mathbf{u}, \boldsymbol{\pi}) = 0$$

States:  $\mathbf{x} = (\psi, \theta_i, \epsilon, \alpha)^T$

Inputs:  $\mathbf{u} = (\beta_i, \tilde{T}_{elc})^T$   $\tilde{T}_{elc} = \frac{T_{elc}}{1/2\rho ARV^2}$

Non-dimensional parameters

$$\boldsymbol{\pi} = \left( \text{Ma}, \text{Re}, \text{Fr}, \text{Lo}, \lambda, \tau, \frac{\omega_\alpha}{\Omega}, \frac{\omega_{\theta_i}}{\omega_\alpha}, \frac{\omega_\epsilon}{\omega_\alpha} \right)^T$$



# Wind Turbine Dimensional Analysis

**Non-dimensional parameters** of physical relevance resulting from dimensional analysis:

- Mach:  $Ma = V/a$  Effect typically negligible for WTs
- Reynolds:  $Re = \rho V c / \nu$  Inertial/viscous aerodynamic force ratio
- Froude:  $Fr = V^2 / gR$  Aerodynamic/gravitation force ratio
- Lock:  $Lo = C_{L\alpha} \rho c R^4 / J_\theta$  Aerodynamic/inertial force ratio
- TSR:  $\lambda = \Omega R / V$
- Non-dim. time:  $\tau = \Omega t$
- Non-dim. tower freq.:  $\tilde{\omega}_\alpha = \omega_\alpha / \Omega, \quad \omega_\alpha = \sqrt{K_\alpha / J_\alpha}$
- Flapping freq. placement:  $\omega_{\theta_i} / \omega_\alpha, \quad \omega_{\theta_i} = \sqrt{K_{\theta_i} / J_{\theta_i}}$
- Shaft freq. placement:  $\omega_\epsilon / \omega_\alpha, \quad \omega_\epsilon = \sqrt{K_\epsilon / J_\epsilon}$



# Size Effects on Wind Turbines

Consider two wind turbines  $\mathbb{M}$  and  $\mathbb{P}$  with scale ratio

$$n = R_{\mathbb{M}}/R_{\mathbb{P}}$$

operating in the **same wind**

$$V_{\mathbb{M}} = V_{\mathbb{P}}$$

at the **same TSR**

$$\lambda_{\mathbb{M}} = \lambda_{\mathbb{P}}$$



# Size Effects on Wind Turbines

Quantity	Symbol	Scaling coefficient	Comment
Rotor speed	$\Omega_M/\Omega_P$	$n^{-1}$	
Reynolds	$Re_M/Re_P$	$n$	
Froude	$Fr_M/Fr_P$	$n^{-1}$	Gravity important only for very large sizes
Lock	$Lo_M/Lo_P$	$n^0$	Assuming same density
Non-dim. freq.	$\tilde{\omega}_{\alpha M}/\tilde{\omega}_{\alpha P}$	$n^0$	Assuming same Young modulus (*)
Freq. placements	$(\omega_{\theta_i}/\omega_{\alpha})_M/(\omega_{\theta_i}/\omega_{\alpha})_P$ $(\omega_{\epsilon}/\omega_{\alpha})_M/(\omega_{\epsilon}/\omega_{\alpha})_P$	$n^0$	
Power	$P_M/P_P$	$n^2$	
Torque	$Q_M/Q_P$	$n^3$	
Bending stress	$\sigma_M/\sigma_P$	$n^0$	Size indep. stress level
Shaft shear stress	$\tau_M/\tau_P$	$n^0$	Size indep. stress level

- **Aeroelastic effects unchanged** (Lock and frequencies), except for possible influence of Froude (only large wind turbines)
- **Stress level unchanged** (except for gravity induced loads)

(\*) Using a more realistic beam-like natural frequency  $\omega = \sqrt{EJ/(mL^4)}$



# Scaling Laws for Wind Turbine Models

## General scaling procedure:

- Given **scale factor**  $n = R_M/R_P$
- Find **velocity**  $n_V = V_M/V_P$  and **time**  $n_t = t_M/t_P$  scalings

## Enforce:

- Time:

$$\tau_M = \tau_P \quad (\Omega t)_M = (\Omega t)_P \quad \longrightarrow \quad n_\Omega = \Omega_M/\Omega_P = 1/n_t$$

- Tip-speed-ratio:

$$\lambda_M = \lambda_P \quad (\Omega R/V)_M = (\Omega R/V)_P \quad \longrightarrow \quad n_V = n_\Omega n = n/n_t$$

- Lock:

$$L_{O_M} = L_{O_P} \quad (C_{L_\alpha} \rho c R^4 / J_\theta)_M = (C_{L_\alpha} \rho c R^4 / J_\theta)_P \quad \longrightarrow \quad \rho_{m_M} / \rho_{m_P} = 1$$

Remark: Lock can always be fixed with material density



# Scaling Laws for Wind Turbine Models

**Enforce** (continued):

- Frequency placement:

$$\tilde{\omega}_{\alpha\text{M}} = \tilde{\omega}_{\alpha\text{P}} \quad (EJ/(mL^4\Omega^4))_{\text{M}} = (EJ/(mL^4\Omega^4))_{\text{P}}$$

$$\rightarrow (EJ)_{\text{M}}/(EJ)_{\text{P}} = n^6/n_t^2$$

Remark: frequency placement can always be fixed with stiffness

**Resulting errors:**

- Reynolds  $\text{Re}_{\text{M}}/\text{Re}_{\text{P}} = n^2/n_t$
- Froude  $\text{Fr}_{\text{M}}/\text{Fr}_{\text{P}} = n/n_t^2$
- Mach  $\text{Ma}_{\text{M}}/\text{Ma}_{\text{P}} = n/n_t^2$

**Important remark:** only unknown left is time scaling  $n_t$



# Scaling Laws for Wind Turbine Models

**Optimal scaling:** minimize **Reynolds error** (reduce airfoil aerodynamic differences) + **scaled time acceleration** (reduce active control frequency)

$$\min_{n_t} (k^2 \text{Re}_M / \text{Re}_P + t_M / t_P) = \min_{n_t} (k^2 n^2 / n_t + n_t)$$

which gives **mismatched Mach scaling**  $n_V = 1/k$

Quantity	Symbol	Scaling coefficient	Comment
Rotor speed	$\Omega_M / \Omega_P$	$1/(nk)$	
Reynolds	$\text{Re}_M / \text{Re}_P$	$n/k$	
Froude	$\text{Fr}_M / \text{Fr}_P$	$1/(nk^2)$	Gravity important only for very large sizes
Lock	$\text{Lo}_M / \text{Lo}_P$	$n^0$	Implies same density
Non-dim. freq.	$\tilde{\omega}_{\alpha M} / \tilde{\omega}_{\alpha P}$	$n^0$	Implies bending stiffness ratio
Power	$P_M / P_P$	$n^2 / k^3$	
Torque	$Q_M / Q_P$	$n^3 / k^2$	
Material density	$\rho_{mM} / \rho_{mP}$	$n^0$	Enforces Lock constraint
Bending stiffness	$(EJ)_M / (EJ)_P$	$n^4 / k^2$	Enforces freq. constr.



# Scaling Laws for Wind Turbine Models

**Example:** scaling a large wind turbine ( $R_P \approx 30$  m) to fit in a 4m x 4m test section ( $R_M \approx 1$  m):

$$n \approx 1/30 \quad k = 2$$

Quantity	Symbol	Scaling coefficient
Rotor speed	$\Omega_M / \Omega_P$	15
Reynolds	$Re_M / Re_P$	1/60
Froude	$Fr_M / Fr_P$	7.5
Lock	$Lo_M / Lo_P$	1
Non-dim. freq.	$\tilde{\omega}_{\alpha M} / \tilde{\omega}_{\alpha P}$	1
Power	$P_M / P_P$	1/57600
Torque	$Q_M / Q_P$	1/432e3
Material density	$\rho_{mM} / \rho_{mP}$	1
Bending stiffness	$(EJ)_M / (EJ)_P$	1/1296e4

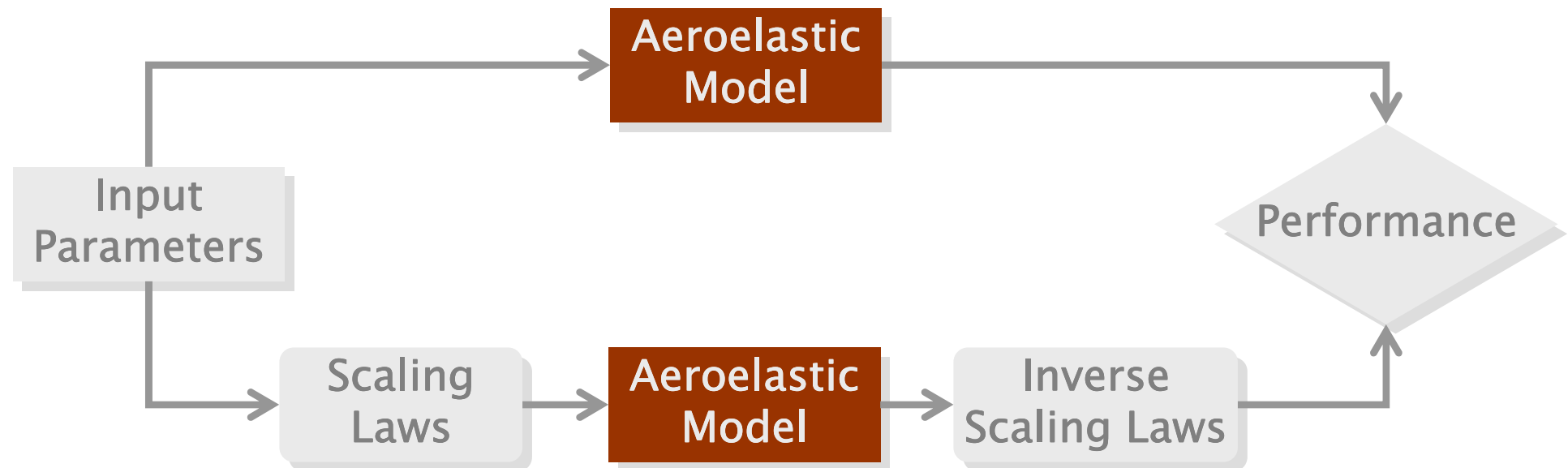
- **Aeroelastic effects unchanged** (Lock and frequencies), except for possible influence of Froude (only large wind turbines)
- Moderate **Reynolds mismatch**, can be partially mitigated with transition strips or similar devices
- **Higher required control frequencies** than on full scale system, but manageable with sufficient computing power



# Testing of Scaling Laws

## Approach:

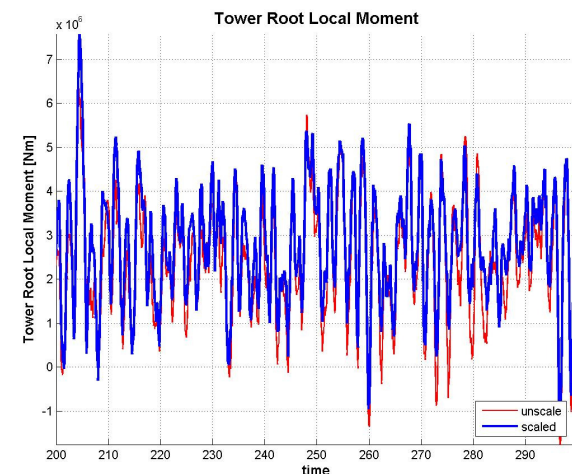
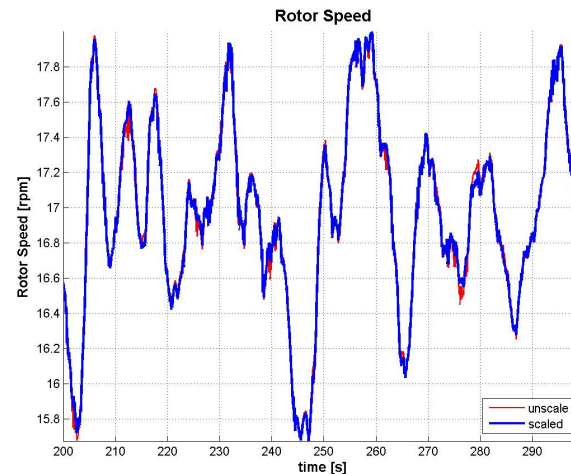
- Choose comparison metrics (e.g., fatigue damage index in turbulent winds, load peak values for gust response, etc.)
- Simulate response of scaled and full-scale models
- Compare responses upon back-scaling of scaled results using metrics



# Testing of Scaling Laws

**Example:** performance comparison of two control laws on full scale and Mach-mismatched-scaled model

Turbulent 16 m/s wind  
 Re full scale  $5.25e+6$   
 Re scaled  $4.6e+5$



“Goodness” of one controller wrt the other is practically the same when tested on full scale and scaled model ▶

**Hence:**

Scaled model is appropriate for conducting control law comparisons

