Computational dynamics

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ABSTRACT

The simulation of aerospace systems leads to multidisciplinary problems dealing with the modeling of complex flexible mechanisms interacting with various other coupled fields. Structural models of the elastic bodies are connected together by mechanical joints, and coupled to aerodynamic models, to hydraulic and electromechanical sub-systems, engine, sensors and other elements. Modern multibody dynamics technology, based on a combination of linear and non-linear finite element methods, techniques for enforcing constraints, coupled simulation methods and accessory algorithmic tools, is capable of delivering models of high sophistication for a wide range of highly demanding industrial applications in aerospace engineering.

KEY WORDS: multibody dynamics; differential algebraic equations; coupled multi-field problems

1. INTRODUCTION

The dynamic analysis of aerospace systems requires the ability to model flexible structural mechanisms made of several components in relative motion with respect to one another, connected by various kinds of mechanical linkages, assembled in topological configurations of arbitrary complexity. Such structural elements are typically interacting with multiple coupled fields, including aerodynamics and those arising in the modeling of hydraulic and electrical components, engines and other actuators, possibly experiencing contacts and impact events among the bodies of the system and with the environment, often operating in closed-loop under the action of control systems. The landing gear of figure 1 is one such flexible mechanism, which undergoes large motions, is coupled to hydraulic and electromechanical systems, and experiences aerodynamic and contact loads.

These complex multi-field models find applicability in a multitude of tasks related to the design, verification and simulation of aerospace systems, including the evaluation of loads in critical components, the analysis of vibratory levels, the estimation of a variety of performance indices, the assessment of the handling qualities of a vehicle, the synthesis of control laws, the training of operators, etc. Some of these activities blend together classical disciplines such as structural dynamics, aero-servo-elasticity and flight mechanics, with an ever increasing demand from industry of the fidelity and reliability of the simulations.

In recent years, multibody dynamics has emerged as the technology of choice for addressing these simulation needs. In fact, the success of modern multibody analysis systems can be traced to their ability to model mechanisms by assembling generalized elements from within a library, much like in classical finite element methods (FEM). Each element provides a basic functional building block, such as for example a rigid body, a hinge, a motor, an interacting field embedded in an external application, etc. By assembling and connecting such generalized elements together, one has a powerful way of describing complex systems with the desired level of modeling fidelity. This also opens the way to hierarchical modeling within the same approach, where more or less refined models of the same system are built based on required accuracy, the need to resolve or not certain scales in the solution, the limitation of computational cost or the maximization of execution speed, or various possible tradeoffs.

Key to this modeling approach is the capability of handling constraints, which in turn affects the choice of coordinates, the writing of the governing equations and their integration in time, the coupling of the multiple interacting fields, and a number of related algorithmic details. Several ad-hoc formulations have been developed in the multibody dynamics literature for niche applications: for example, methods have been developed for maximum execution speed in the analysis of systems in tree-like topologies, or for handling efficiently crash analysis simulations using specialized equivalent models of the contacting parts. While some of these niche formulations find applicability in the aerospace field, in this work we focus exclusively on methods which can be used for modeling complex flexible systems with arbitrary topological configurations, such as those who may be used for the development of comprehensive models of vehicle systems or sub-systems.

The modeling of rotary wing vehicles embodies the challenges that need to be faced in the simulation of aerospace systems. Figure 2 shows the idealized model of a conventional articulated rotor system: the blades, made of composite materials and experiencing geometric stiffening effects due to their rotation at high angular rates, are actuated by a complex system of control linkages and are connected to the hub by hinges and hydraulic dampers. The figure shows the modeling of the rotor system by means of rigid bodies, beam elements, mechanical joints, aerodynamic and hydraulic elements. In some applications, one may want to include, according to the modeling needs, a model of the elastic fuselage, of the tail rotor, of the landing gears, of the drive-train, or of other components. Bearingless multiple load-path configurations are also common, as the use of elastomeric dampers, gimballed rotors, and many other possible configurations, radically different from the one of the figure.

Historically, the modeling needs of each rotorcraft manufacturer were answered by developing dedicated software programs, tailored to a single specific configuration. In time, the need for more sophisticated analysis tools capable of capturing all relevant physical processes and the continuous proposal of new configurations and technological solutions, has pushed the rotorcraft field towards the more general paradigm of multibody dynamics, to the point that most (if not all) current industrial-level general-purpose rotorcraft codes are based on multibody formulations. Using the modular approach of multibody dynamics, one may assemble elements from within a library to describe a given rotorcraft system, and new specialized elements may be included in the library for describing functionalities not yet present. Modern multibody technology is capable of developing highly sophisticated models of rotary wing systems, although the coupled field of aerodynamics remains a critical area where important improvements are still necessary. Clearly, this same paradigm is applicable to other fields of engineering dealing with the modeling of complex systems.

In this work, we first review the derivation of the governing equations of multibody systems in Section 2. The introduction of constraints leads to differential algebraic equations, whose solution needs special precautions and leads to a variety of formulations. Next, we describe in Section 3 some common element models, including body, joint and coupled field models. Section 4 describes the most important solution procedures, which include static and dynamic analyses, as well as some more specialized ones as the solution of optimal control problems which find applicability in the modeling of maneuvers at the boundaries of the operating envelope of vehicles. Finally, Section 5 describes the corollary supporting technologies of parameter estimation and of model reduction. Space limitation preclude an in depth analysis; therefore, some topics are only briefly sketched while others are not covered, such as the discussion on coupling procedures for multi-field problems.

2. EQUATIONS OF MOTION AND SOLUTION TECHNIQUES

2.1. Choice of coordinates

As for many other aspects of multibody dynamics, even the very basic choice of the coordinates used for formulating the equations of motion can be made in several very different ways (Geradin and Cardona, 2001). Clearly, each way has its own specific features, which make it more or less suited to a given application niche.

A classical approach used mainly for analytical developments, is to adopt a minimal set of coordinates. For example, in a planar four-bar mechanism, the motion of each one of the bodies can be expressed in terms of a single coordinate (e.g., the relative angle between two consecutive bars) since the system has one single degree of freedom. This approach leads to a minimum number of unknowns and of equations. However, the equations are typically very complicated and highly non-linear. This approach is also difficult to apply to arbitrary topologies, and it is unsuitable for flexible systems.

Another approach is to use relative, recursive Lagrangian coordinates. For example, again in the planar four-bar mechanism case, the motion of each body can be expressed in terms of the relative angle between that body and the previous one. More in general, the Denavit-Hartenberg method (Geradin and Cardona, 2001) has been developed for describing with a minimum number of coordinates the relative motion of a body connected to another one by a joint, which leaves some degrees of freedom free and constraints the others. Clearly, such an approach is particularly efficient for tree topologies, such as those encountered in robotics applications, where it leads to a minimum number of unknowns and equations, and in fact variants of this approach have been used with success for developing very efficient simulation algorithms for real-time applications. For systems with arbitrary topologies, and in the presence of closed loops, the method becomes much more complicated, and can not avoid the introduction of loop-closure constraints. Furthermore, the resulting equations are usually very complex and highly non-linear, and hence expensive to evaluate. For flexible systems modeled using modal-based elasticity, this approach has been used for expressing the motion of a floating frame of reference, whose purpose is to describe the gross rigid motion of the body; in addition to this motion, the elastic body is assumed to undergo a small deformation about the floating frame (Shabana, 1998). This approach leads again to complex expressions, most notably of the kinetic energy, and does not account for the geometric stiffening effect, which is important in certain applications, as for example those involving rotating blades.

A formulation which is more amenable to problems with arbitrarily complex configurations is the one based on Cartesian coordinates. In this case, each body is regarded as free in space and described with its own set of coordinates. Next, constraints are introduced to represent the presence of mechanical joints among the various bodies of the system. This way, the multibody system is described by a redundant set of unknowns, composed by the collection of the sets of each body, often with the addition of Lagrange multipliers for the enforcement of the constraints (see next Section). This set of unknowns is redundant, as opposed to the previous choices which were minimal or quasi-minimal. On the other hand the equations of motion are much simpler, because they are obtained by assembly of the equations of the individual bodies and of the constraints, with a procedure similar to the one routinely used in finite element analysis. Furthermore, since each body or joint is typically connected with a small number of adjacent bodies or joints, the equations of motion have a high degree of sparsity, and present a banded nature, possibly after a suitable reordering of equations and unknowns; these aspects are crucial for the efficiency of the solution when analyzing large scale problems.

Another important characteristic of a formulation expressed in terms of Cartesian coordinates, is that it is readily amenable to the analysis of flexible systems using the finite element method. In fact, each elastic body in a multibody system may be described in terms of finite element unknowns, i.e. positions and possibly rotations, as in the case of beams and shells, at mesh nodes. Specific nodes in the mesh may then be used for connecting the elastic body with other bodies or joints in the system, again using a standard assembly process. Clearly, the finite element formulation of elastic bodies must be capable of describing in exact terms the kinematics of the body motion. Such formulations are termed geometrically exact, and are based on definitions of the (invariant) strain energy V in terms of strain measures $\boldsymbol{\epsilon}$ which are unaffected by arbitrarily large rigid body motions, i.e. $V(\boldsymbol{\epsilon}) = V(\boldsymbol{R}\boldsymbol{\epsilon})$ if $\boldsymbol{R}$ is an arbitrary rotation (Borri, Trainelli and Bottasso, 2000; Geradin and Cardona, 2001; Bauchau, Bottasso and Trainelli, 2003). Contrary to the floating frame approach, these formulations correctly account for geometrically non-linear effects due to the kinematics being exact (finite rotation approach), and lead to particularly simple expressions of the inertial forces.

2.2. Lagrange equations and the index 3 form

The Lagrangian of a multibody system $\mathcal{M}$ can be written as

$$L^* = \sum_i L_i(\dot{\boldsymbol{q}}, \boldsymbol{q}) + \boldsymbol{\lambda} \cdot \boldsymbol{c} + \frac{1}{2}\,\rho\,\boldsymbol{c} \cdot \boldsymbol{c}, \tag{1}$$

where $L_i(\dot{\boldsymbol{q}}, \boldsymbol{q})$ is the Lagrangian of the ith body in the system, and $\boldsymbol{q}$ are generalized coordinates. The system is subjected to constraints

$$\boldsymbol{c} = 0, \tag{2}$$

which are enforced through two modifying terms of the Lagrangian: the first, $\boldsymbol{\lambda}\cdot\boldsymbol{c}$, uses Lagrange multipliers $\boldsymbol{\lambda}$, while the second, $1/2\,\rho\,\boldsymbol{c}\cdot\boldsymbol{c}$ is a penalty-like term with penalty factor ρ. Clearly, both terms are null when the solution satisfies the constraints.

For $\boldsymbol{\lambda} = 0$ one obtains a pure penalty formulation. To enforce the presence of the constraints, it is necessary to choose ρ as a large number since $\boldsymbol{c} \to 0$ only if $\rho \to \infty$; this however leads to ill-conditioning of the problem, so that this practice is not recommended for the large scale complex applications which are the focus of the present work.

For $\rho = 0$ one obtains a purely Lagrangian formulation, which increases the number of unknowns with respect to the penalty approach, since one has now to solve for both the coordinates $\boldsymbol{q}$ and the multipliers $\boldsymbol{\lambda}$. The multipliers, however, rigorously account for the presence of the constraints and their reactions on the bodies of the system.

The formulation retaining both terms, equation (1), is termed the augmented Lagrangian approach. Since constraints are enforced by Lagrange multipliers, ρ does not need to be chosen as a large number, and hence one avoids the resulting ill-conditioning. Nonetheless, the penalty-like term proves to be useful because it allows one to factorize the system Jacobian (as required for the solution of the equations using a Newton-like method) without pivoting (Bauchau, Epple and Bottasso, 2009), which is crucial for efficiency when dealing with large scale problems by retaining the bandedness of the iteration matrix.

Imposing the stationarity of L^*, one obtains the equations of Lagrange:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{A}^T\boldsymbol{\mu} = \boldsymbol{g}, \tag{3a}$$

$$\boldsymbol{c} = 0, \tag{3b}$$

where $\boldsymbol{\mu}$ are augmented Lagrange multipliers, $\boldsymbol{\mu} = \boldsymbol{\lambda} + \rho\boldsymbol{c}$, and $\boldsymbol{A}^T\boldsymbol{\mu}$ are the constraint reactions. Clearly, at convergence when $\boldsymbol{c} = 0$ then $\boldsymbol{\mu} = \boldsymbol{\lambda}$, and the penalty-like term has no effect on the computed solution. If the constraints are holonomic, i.e. $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{q})$, then $\boldsymbol{A} = \partial\boldsymbol{c}/\partial\boldsymbol{q}$ and $\boldsymbol{A}$ is termed the constraint Jacobian; if the constraints are non-holonomic, i.e. $\boldsymbol{c} = \boldsymbol{c}(\dot{\boldsymbol{q}}, \boldsymbol{q})$, then they are invariably linear in the velocities $\dot{\boldsymbol{q}}$ for virtually all practical applications in mechanics and $\boldsymbol{c} = \boldsymbol{A}\dot{\boldsymbol{q}} + \boldsymbol{a}$.

Equations (3) are differential algebraic equations (DAEs), since they have among their unknowns the algebraic variables $\boldsymbol{\lambda}$. These equations can be turned into ordinary differential equations (ODEs) by taking three analytical differentiations of equation (3b), which have the effect of introducing terms in $\dot{\boldsymbol{\lambda}}$ and therefore eliminate the algebraic nature of the problem. For this reason, equations (3) are said to be in index 3 form (Hairer and Wanner, 1996).

2.3. Transformation into ODEs and DAE index reduction approaches

It was recognized early in the literature that the solution of high index DAEs can cause severe numerical difficulties. For example, the analysis of Petzold and Lötstedt, 1986 demonstrated that the condition number of the iteration matrix for the index 3 form of equations (3) is $\mathcal{O}(h^{-3})$. Furthermore, it was shown that errors grow in the Lagrange multipliers as $\mathcal{O}(h^{-3})$, and at lower but still unfavorable rates in the displacement and velocity fields. Therefore, for DAEs one obtains a rather surprising behavior: the solution, which should converge to the true one as the time step size vanishes, on the contrary becomes polluted by the unavoidable errors which are due to the use of finite precision arithmetic and of finite convergence tolerances for arresting Newton iterations. Hence, contrary to intuition, mesh refinement makes the problem harder to solve. This unusual behavior manifests itself when using small time steps, which on the other hand are often necessary for achieving the desired level of accuracy or for resolving fast solution components present in the solution, for example when analyzing contact/impact phenomena.

The multibody dynamics literature abounds with methods for turning problem (3) into an ODE one or for reducing the index from 3 to 2 or 1. A comprehensive treatment of the methods which have been proposed goes beyond the scope of this work and would be incompatible with the current space limitations. The interested reader may however consult the review offered in Laulusa and Bauchau, 2008 and Bauchau and Laulusa, 2008, which also contain a rich list of relevant references.

Very synthetically, a first family of methods formulates the problem in ODE terms by using a minimum set of unknowns. This can be accomplished at the level of the coordinates, or at the level of the velocities.

Considering the first of these two options, if the system is described in terms of n (redundant) coordinates $\boldsymbol{q}$ and has m constraints, it is conceptually possible to split the coordinate vector into an independent set of $n - m$ coordinates $\boldsymbol{q}_i$ and a dependent set $\boldsymbol{q}_d$ of size m, whereby the dependent variables can be expressed in terms of the independent ones: $\boldsymbol{q}_d = \boldsymbol{q}_d(\boldsymbol{q}_i)$. This leads to a formulation of the equations of motion without Lagrange multipliers, and hence in ODE form.

Similarly, working at the level of velocities, the so called Maggi-like methods seek to write the generalized velocities $\dot{\boldsymbol{q}}$ in terms of a minimal set of $n - m$ generalized speeds $\boldsymbol{e}$

$$\dot{\boldsymbol{q}} = \boldsymbol{B}\boldsymbol{e} + \boldsymbol{b}, \tag{4}$$

while accounting for the fact that generalized velocities are constrained as

$$\boldsymbol{A}\dot{\boldsymbol{q}} + \boldsymbol{a} = 0. \tag{5}$$

This is true by definition in the case of non-holonomic constraints, or it is obtained by differentiating holonomic constraints once with respect to time, i.e. $\dot{\boldsymbol{c}} = 0$. It may be proven that $\boldsymbol{B}$ spans the null space of $\boldsymbol{A}$, i.e. $\boldsymbol{A}\boldsymbol{B} = 0$. Hence, premultiplying (3a) by $\boldsymbol{B}^T$ effectively eliminates the Lagrange multipliers, leaving an ODE in the generalized speeds $\boldsymbol{e}$ and accelerations $\dot{\boldsymbol{e}}$.

Both methods suffer from several drawbacks, which make them unsuitable for the large scale FEM applications which are the focus of the present work, although they have found their niches of applicability in other sectors of multibody dynamics. First of all, both when working at the level of coordinates and at the level of velocities, the choice of the minimal set is not unique. This means that one has to devise an automated way of picking a set, typically through some form of optimality. Furthermore, the definition of the set is often local, which means that the set might become ill-defined in certain configurations of the system; this means that the set must be continuously monitored and updated throughout the simulation. While this is certainly possible, it significantly complicates the implementation. Furthermore, while equations in the redundant form (3) are typically highly sparse and present a banded pattern, equations in minimal form are usually dense or with limited sparsity; unfortunately, the loss of bandedness incurs in very significant computational costs for large scale FEM-based models. Hence, for large problems, the potential advantage of having lowered the number of unknowns by moving from a redundant set with multipliers to a minimal set, is offset by the increased complexity of the equations. Finally, in the case of Maggi's methods, holonomic constraints are accounted for through their time derivatives $\dot{\boldsymbol{c}}$, which will cause the position-level constraints to drift away from the manifold, i.e. in general the numerical solution $\boldsymbol{q}_h$ will be such that $\boldsymbol{c}(\boldsymbol{q}_h) \neq 0$ (although this drift is often limited and sufficiently small for being acceptable in practical situations).

A second class of methods is based on the lowering of the index of problem (3), with the bulk of formulations focusing on the index 1 form obtained by appending to equation (3a) the second derivative of the constraints, i.e. $\ddot{\boldsymbol{c}} = 0$, which yields

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{A}^T\boldsymbol{\mu} = \boldsymbol{g}, \tag{6a}$$

$$\boldsymbol{A}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{A}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{a}} = 0. \tag{6b}$$

The Lagrange multipliers can be readily eliminated from system (6) either by straightforward substitution (compute $\ddot{\boldsymbol{q}}$ from (6a), insert into (6b), solve in terms of $\boldsymbol{\lambda}$ and replace $\boldsymbol{\lambda}$ into (6a)), or by using a Moore-Penrose generalized inverse (Laulusa and Bauchau, 2008). Yet another alternative is to compute the null space $\boldsymbol{B}$ of $\boldsymbol{A}$, and premultiply equation (6a) by $\boldsymbol{B}^T$ to eliminate the constraint reactions by exploiting the fact that $\boldsymbol{AB} = 0$.

Even this second class of methods is affected by several problems. First, here again the manipulations of the equations which are necessary for turning the system in index 1 form and for subsequently solving it, irremediably destroy the banded sparsity of the matrices and render the equations highly involved and complicated. Furthermore, the constraints are now enforced at the acceleration level, instead of the position one. It can be shown that the acceleration-level constraint is unstable, so that in this case the solution will experience a much larger drift from the constraint manifold than in the case of the velocity-level enforcement. Since this is in most cases unacceptable, various techniques have been developed for improving the satisfaction of the constraints. Here again details go beyond the scope of the present work, but in synthesis it can be said that most approaches are based on one of the following ideas: the first is to modify the acceleration constraint so as to make it stable by the addition of the velocity and position level constraints as damping-like and stiffness-like terms, respectively; the second is based on the projection of the solution back onto the constraint manifold (Bauchau and Laulusa, 2008). The former can not guarantee the exact satisfaction of the constraint conditions but only aims at the stabilization of the drift effect, while the latter one can indeed compute solutions compatible with the constraints at machine accuracy or assigned levels of tolerance.

As a notable exception to the situation described above, a recent formulation has been proposed based on the null space approach (Betsch and Leyendecker, 2006). In this case the null space is computed starting from a discrete (as opposed to the continuous, as usually done) version of the equations of motion, as obtained by applying a conserving time discretization scheme. Furthermore, by exploiting the simpler nature of the discrete equations, the contribution to the null space of each individual constraint condition can be computed explicitly and analytically up-front, which results in a major simplification of the numerical procedure.

2.4. Direct solution of index 3 DAEs and the importance of scaling

As previously explained, much of the literature dealing with various ways to solve problem (3), does not address the pollution problem directly, but rather tries to avoid the problem by turning the problem into an ODE one or by lowering the index. It was argued that, unfortunately, this strategy incurs in other problems since constraints, being imposed at the velocity or acceleration level rather than at the displacement one, typically drift away from the constraint manifold. This in turn calls for additional corrective actions, in the form of constraint stabilization and/or projection back onto the manifold. An alternative approach is to re-write the governing equations so as to include both the position-level and velocity-level constraints (Gear, Leimkuhler and Gupta, 1985; Borri, Trainelli and Croce, 2006), which however comes at the cost of additional problem unknowns.

What is even more important for the applications of interest here, is that all these approaches are unsuitable for FEM-based applications, due to the loss of banded structure of the problem and to the complexity and numerical cost of the manipulations of the equations. From this point of view, it is clear that the redundant but highly sparse, algebraic formulation expressed by equations (3) is highly preferable, as recognized already by several authors (Orlandea, Chace and Calahan, 1977; Cardona, 1989). Furthermore, since the index 3 form enforces the constraints at the position level, the "inherited" velocity and acceleration-level constraints, although not explicitly accounted for, will be approximatively satisfied since they are obtained by differentiation of the enforced ones; as noted above, the opposite is not true for the lower index formulations, where drift effects are present since "inherited" constraints are obtained by integration and not differentiation. Therefore, by adopting an index 3 formulation one avoids all complications associated with constraint stabilization and projection, while working with relatively simple equations with a banded structure.

Yet, the problem of numerical pollution has to be solved in order to arrive to fully effective computer implementations. Stiff integrators denoted by high frequency numerical damping help in this regard (Orlandea, Chace and Calahan, 1977) and are also useful for other reasons, as noted below. But it is only recently that the pollution problem has been tackled directly (Bottasso, Bauchau and Cardona, 2007; Bottasso, Dopico and Trainelli, 2008; Bauchau, Epple and Bottasso, 2009), leading to a remarkably simple solution of the problem, based on an early hint reported but not analyzed in Cardona, 1989. The idea is to scale the problem so as to eliminate the unfavorable behavior with respect to the time step length h. To this effect, one defines a non-dimensional time $\tau = t/h$ and works with derivatives with respect to τ instead of t, which are indicated here as $\mathrm{d}(\cdot)/\mathrm{d}\tau = (\cdot)'$. This way, the time step in non-dimensional time becomes of unit length. Of course, this scaling of time can be defined at each time step based on the current value of h, so that the procedure can be applied with no difficulty in the case of variable time step sizes.

As for most problems in mechanics, a further improvement of the overall conditioning of a numerical process can be obtained by using unknowns which are well scaled, i.e. all roughly of $\mathcal{O}(1)$. To this effect, one may use, in addition to the scaling of time, also non-dimensional coordinates $\bar{\boldsymbol{q}}$ instead of the dimensional ones, $\boldsymbol{q}$. Similarly, non-dimensional functions of the non-dimensional coordinates are noted in the following as $\overline{(\cdot)}$.

This way, a non-dimensional augmented Lagrangian may be written as

$$\bar{L}^* = \sum_i \bar{L}_i(\bar{\boldsymbol{q}}', \bar{\boldsymbol{q}}) + h^2 \boldsymbol{\lambda} \cdot \bar{\boldsymbol{c}} + \frac{1}{2}\, \rho h^2\, \bar{\boldsymbol{c}} \cdot \bar{\boldsymbol{c}}. \tag{7}$$

Notice that, since the Lagrangians $L_i(\dot{\boldsymbol{q}}, \boldsymbol{q})$ are quadratic in the velocities, the introduction of the non-dimensional time has caused the appearance of a term h^2 in the two augmenting terms, which must now be carefully analyzed.

For the first of the two, the idea is to define $h^2\boldsymbol{\lambda}$ as the scaled Lagrange multiplier of the new problem, i.e. we set $\bar{\boldsymbol{\lambda}} = h^2\boldsymbol{\lambda}$ and we solve for $\bar{\boldsymbol{\lambda}}$ instead of $\boldsymbol{\lambda}$. Once a solution has been computed, $\boldsymbol{\lambda}$ can be straightforwardly recovered as $\boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}}/h^2$, so that constraint reactions can be evaluated.

For the second of the two, it is necessary to recognize that, for the effect of the penalty-like term not to vanish with $h \to 0$, one must set $\rho h^2 = r$, where r is now the penalty-like coefficient of the new scaled formulation.

With these choices, by imposing the stationarity of the non-dimensional augmented Lagrangian $\bar{L}^*$, one gets the scaled equations of motion of multibody systems in index 3 form:

$$\bar{M} q'' + \bar{A}^T \bar{\mu} = h^2 \bar{g}, \tag{8a}$$

$$\bar{c} = 0, \tag{8b}$$

where $\bar{\mu}$ are scaled augmented Lagrange multipliers, $\bar{\mu} = \bar{\lambda} + r\bar{c}$. It may be proven (Bauchau, Epple and Bottasso, 2009), that the scaled problem (8) is now completely insensitive to the pollution which affects (3). In fact, the error propagation rates in the solution fields and the condition number are all $\mathcal{O}(h^0)$, which is what one would expect from the solution of an ODE. Hence, by scaling alone one may eliminate the problems associated with small time step sizes of multibody DAEs, and make them not harder to treat than standard ODEs.

It should be noted that this result is achieved at the sole price of working in non-dimensional time with an augmented Lagrangian formulation, which means that one has to recover the Lagrange multipliers by scaling them back, a trivial scalar operation of negligible cost. It should be further remarked that, typically, further improvements in the conditioning and robustness of the problem may be obtained by working with well scaled (non-dimensional) variables and equations, as for most numerical problems in mechanics, although this will not cure by itself the unfavorable behavior with respect to h of the original problem; as previously noted, this can only be eliminated by introducing the non-dimensional time and h^2-scaled Lagrange multipliers.

2.5. Numerical integration techniques

The numerical integration of the equations governing high index DAEs has received considerable attention in the literature. The solution of equations (3) (or (8)) presents features which are in common with non-linear transient finite element applications. Specifically, since the equations typically model the low-medium frequency range of aero-servo-elastic systems, implicit schemes are superior to explicit ones, which on the other hand would be more effective for the simulation of processes with very fast dynamic scales as in case of crash simulations. Furthermore, it is a well known fact that the higher frequency content of FEM models does not accurately represent the behavior of the true system, and it is in fact an artefact of the discretization process. When the response of these higher frequencies is excited, noise of a numerical origin affects the solution, a problem which may be exacerbated in the presence of non-linearities to the point of leading to the blow-up of the computation. Therefore, stiff time integration schemes are typically used, i.e. integrators which act as low pass filters for the lower accurate modes and which damp out the higher unphysical ones (Hairer and Wanner, 1996; Geradin and Cardona, 2001), typical examples being the dissipative members of the Newmark family of schemes (and the related modified-α method), or stiffly accurate Runge-Kutta schemes.

The idea of designing integration schemes for which non-linear proofs of stability are possible, has also been pursued with success. Dot multiplying equations (3a) by $\dot{q}$, it may be easily proven that for scleronomic (time indipendent, i.e. $\partial c/\partial t = 0$) constraints, one has

$$\dot{E} = P_e, \tag{9}$$

which states that the time rate of change of the total mechanical energy E of the system is equal to the power generated by the external forces, P_e. This implies that constraint reaction forces do not generate nor absorb power, i.e.

$$\dot{q} \cdot A\lambda = 0. \tag{10}$$

Energy methods are then based on the idea of allowing for the proof of these two facts, equations (9) and (10), at the discrete solution level. The procedure is based in essence on two steps: 1) for each unconstrained body model in the multibody system, one devises a temporal scheme for which it is possible to prove that the discrete rate of change of energy within a time step is equal to the algorithmic power generated by the external forces; 2) for each joint model in the multibody system, one devises an algorithmic discrete version of the constraint reactions for which it is possible to prove that no power is generated or absorbed within a time step. Once such discretizations have been defined for all body and joint models in the system, the assembly of an arbitrary number of such models into a multibody system will imply the existence of a discrete version of equation (9), i.e.

$$\frac{E_{i+1} - E_i}{h} = P_{e_h}, \tag{11}$$

which leads to the notion of unconditional stability of the integrator in the non-linear regime (see Geradin and Cardona, 2001; Bauchau, Bottasso and Trainelli, 2003; Betsch and Leyendecker, 2006 and references therein).

Clearly, energy preserving schemes are unable to damp out the high frequency unphysical modes in the system, by their very design. Hence, a generalization of this concept has been devised which is based on the proof of an energy decay (rather than conservation) statement within each time step; this way one retains non-linear stability since energy is bounded from above, while at the same time achieving the desired goal of removing the unresolved modes from the solution (Bauchau, Bottasso and Trainelli, 2003).

3. ELEMENT MODELS

3.1. Body models

A general-purpose multibody code includes a library of body models, the simplest being a rigid body which can be used for modeling components whose flexibility can be neglected or for introducing localized masses.

The inclusion of flexible elements in a multibody formulation is a very ample subject, which offers a wide range of possibilities both in terms of mathematical models of the body and of associated algorithms. Here only a very short summary of some important aspects of this topic is offered.

Beam models have attracted a great deal of attention in the literature, with the more sophisticated formulations being devoted to the modeling of rotor blades; clearly, beam models also find applicability in the modeling of other slender structural members which can be found on vehicles, such as transmissions shafts, wings, pitch links in a rotorcraft hub system, etc.

The problem of blade modeling is particularly challenging, since it must be possible to represent shearing deformation effects, the offset of the center of mass and of the shear center from the beam reference line, and all the elastic couplings that can arise from the use of tailored composite materials. To provide for accurate modeling at affordable computational costs, the three-dimensional elasticity problem is split into two sub-problems (Giavotto *et al.*, 1983). The first problem is a linear, two-dimensional problem defined over the beam cross-section which provides the sectional elastic constants. The problem is solved by using a linear 2-D finite element approach, where a mesh is used for describing with all necessary details the geometric and material characteristics of the blade cross section. The second problem is a classical non-linear, one-dimensional problem defined along the beam reference line that predicts the non-linear response of the structure when subjected to time dependent loads; this problem is handled in multibody codes using a geometrically exact formulation. At the post-processing stage, recovery relations provided by the 2-D analysis step can be used for computing the three-dimensional displacement, strain and stress fields in the beam in terms of the generalized one-dimensional strain measures computed using the geometrically exact model. The splitting of the three-dimensional problem into two- and one-dimensional parts results in very significant savings in computing time with respect to a standard three-dimensional finite element analysis. Similar approaches have been developed for composite shell models, where a one-dimensional through-the-thickness analysis provides the elastic constants, which are then input in a classical geometrically exact shell model.

Using the Cartesian coordinate approach, as previously noted, it is straightforward to include in a finite element formulation a generic FEM-based body model, as long as the formulation is invariant with respect to rigid body motions. Such models can be used for representing bodies for which the blade or shell assumptions are not valid, as for example it is the case in certain rotor hub systems which may present complex three-dimensional elements and/or may use components characterized by special material properties (e.g., elastomers).

In a vehicle there are often very complex structural components which can be modeled in linear terms, as for example the fuselage of a helicopter. In fact, the vibratory response of the fuselage couples to the main and tail rotor dynamics, affecting the vehicle flight mechanics, the hub loads, etc. An effective way of including the effects of structural members of the level of complexity of a fuselage, is to use a modal based approach. This way a few of the lower modes are extracted from possibly very detailed FEM models of the structure. Next, the modal representation of the structure is included in the multibody model using a component mode synthesis approach, whereby modal amplitudes are used for describing the linear elastic response of the body in a suitably defined moving frame, while boundary degrees of freedom are retained as additional unknowns in the model so as to allow its coupling to the rest of the multibody system (Bauchau and Rodriguez, 2003).

Finally, most general-purpose codes include other specialized body models, as for example wheel models, which can also be seen as suitable combinations of rigid or flexible bodies coupled to contact conditions and contact force models.

3.2. Joint models

Multibody systems are characterized by the presence of joints that impose constraints on the relative motion of the bodies of the model. Most joints used for practical applications can be modeled in terms of the so called lower pairs (Geradin and Cardona, 2001): the revolute, prismatic, screw, cylindrical, planar and spherical joints.

The kinematics of lower pair joints can be described in terms of Cartesian frames. On the (rigid or flexible) body A, we consider a frame with origin at a point on the body whose position vector is $\boldsymbol{u}^A$, denoted by a triad of unit vectors $\mathcal{A} = (\boldsymbol{e}_1^A, \boldsymbol{e}_2^A, \boldsymbol{e}_3^A)$. Similarly, on body B, a frame has origin in $\boldsymbol{u}^B$ and a triad $\mathcal{B} = (\boldsymbol{e}_1^B, \boldsymbol{e}_2^B, \boldsymbol{e}_3^B)$. The relative displacement between the two bodies in the direction aligned with the unit vector $\boldsymbol{e}_i^A$ is noted d_i, while θ_i is the relative rotation about the same vector. Table I defines the six lower pairs in terms of the relative displacement and/or rotation components that can be either free or constrained to a null value.

	d_1	d_2	d_3	θ_1	θ_2	θ_3
Revolute	C	C	C	C	C	F
Prismatic	C	C	F	C	C	C
Screw	C	C	$p\,\theta_3$	C	C	F
Cylindrical	C	C	F	C	C	F
Planar	F	F	C	C	C	F
Spherical	C	C	C	F	F	F

Table I. Definition of the six lower pair joints. F: free, C: constrained. For the screw joint, p is the screw pitch.

All lower pair constraints can be expressed by one of the following two equations:

$$\boldsymbol{e}_i^A \cdot (\boldsymbol{u}^A - \boldsymbol{u}^B) - d_i = 0, \tag{12a}$$

$$\cos\theta_i\,(\boldsymbol{e}_j^A \cdot \boldsymbol{e}_k^B) - \sin\theta_i\,(\boldsymbol{e}_k^A \cdot \boldsymbol{e}_k^B) = 0. \tag{12b}$$

The first equation represents a constraint on the relative displacement by setting $d_i = 0$; on the other hand, by regarding d_i as an additional variable, the same equation serves the purpose of defining the unknown relative displacement in the joint along that direction. Similarly, the second equation either constrains the relative rotation if $\theta_i = 0$, or defines the unknown relative rotation θ_i if this is regarded as a free variable.

The explicit definition of the relative displacements and rotations in a joint as additional unknown variables represents an important detail of the implementation. In fact it allows for the introduction of spring, damper and backlash elements in the joints. Furthermore, the time histories of joint relative motions can be driven according to suitably specified time functions, or can be used as inputs or outputs of control elements. When such additional variables are defined, equations(3) take the following form:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{A}^T\boldsymbol{\mu} = \boldsymbol{g}, \tag{13a}$$

$$\boldsymbol{c}(\boldsymbol{q}) = 0, \tag{13b}$$

$$\boldsymbol{d}(\boldsymbol{\nu}, \boldsymbol{q}) = 0, \tag{13c}$$

where the relative displacements/rotations are denoted by the vector of algebraic variables $\boldsymbol{\nu}$, while (13c) group together all defining equations of the kind (12) in the model. The new constraints and algebraic variables are of index 1, since one single derivative of (13c) is necessary for introducing terms in $\dot{\boldsymbol{\nu}}$, so that system (13) is an index 1-3 DAE.

The lower pairs can be generalized to express kinematic constraints at the instantaneous point of contact between flexible bodies in relative motion (Bauchau and Bottasso, 2001).

3.3. Multidisciplinary models

Modern multibody systems for the comprehensive analysis of vehicles include models which represent all other relevant fields interacting with the structural dynamics elements.

Aerodynamic effects are accounted for by using a variety of approaches, which range from CFD, to free wake models, to more specialized approaches. For example, the models routinely applied in rotorcraft applications (Datta and Johnson, 2008) typically make a combined use of lifting lines based on two-dimensional strip theory, each lifting line being attached to a beam and moving accordingly, coupled with dynamic inflow models denoted by their own set of states which are solved together with the multibody DAEs. Such models are capable of providing sufficiently accurate estimates of the aerodynamic effects, at least in certain flight conditions, at computational costs which are compatible with their routine use in an industrial environment, something that rotorcraft CFD is not yet capable of delivering. These models are further improved by the use of a variety of sub-models which account for blade tip losses, radial and unsteady flow, dynamic stall and other effects.

Contact models account for interactions among bodies or with the environment. Two main families of approaches are used: methods which explicitly model the deformation processes which take place in the contact zone, typically based on FEM formulations using detailed meshes of the contacting parts, and methods where such processes are rendered in a simplified, equivalent form. The latter class of methods are based on the combination of a kinematic model with a contact force model. The former, given a mathematical description of the geometry of candidate parts, determines whether and where the parts are in contact and the speed of their relative motion, while the latter, based on the information provided by the kinematic model, provides the interaction forces between the contacting bodies, including friction.

Other coupled models frequently used in applications include electromechanical models, hydraulic models, sensor models, control elements, etc.

4. SOLUTION PROCEDURES

General-purpose multibody systems implement a number of solution procedures, some of which are reviewed next.

The static analysis solves the governing equations obtained by setting all time derivatives to zero, under the action of given static loads, and may be computed by Newton-like methods, often in conjunction with continuation techniques which allow one to incrementally load the structure, considerably easing the convergence process. A useful generalization of the static analysis concept is to include also the case of assemblies of bodies in the system undergoing rigid body motions about a given point at a constant assigned angular rate. The resulting inertial forces, as well as the possible aerodynamic forces generated by the same motion, can be regarded as steady configuration-dependent loads, and the equilibrium configuration of the system can be easily computed by a standard static analysis. This procedure finds applicability, for example, for rapidly computing the deflected configuration of a steadily spinning rotor in axial flow.

Once the static deflected configuration has been computed, one may consider the dynamic behavior of small amplitude perturbations about the equilibrium configuration. This is obtained by first linearizing the dynamic equations of motion, and then extracting the eigenvalues and eigenvectors of the resulting linear system. Due to the presence of gyroscopic effects when rotating parts are present, the eigenpairs are, in general, complex. Stability can be assessed using similar techniques; for example, for systems characterized by periodic coefficients as rotorcraft vehicles, ad hoc implementations of Floquet method can be used for extracting from a multibody model estimates of the damping levels in the least damped modes of the system.

The dynamic analysis solves the non-linear equations of motion for the complete multibody system. The initial condition are taken to be at rest, or those corresponding to a previously determined static or dynamic equilibrium configuration. The equations of motion of the system are integrated, starting from the given initial conditions, under the action of externally applied loads, given driving inputs, or the action of closed-loop control systems which steer the model according to some given criterion. Complex multibody systems often involve rapidly varying responses, which may render the use of a constant time step a computationally inefficient strategy, so that time step size adaptivity is commonly adopted for increased efficiency.

The above procedures are available in most transient codes. In particular, dynamic analysis are usually considered as initial value problems, where the motion is computed under the action of given inputs. Recently, a more general class of transient solution procedures has been considered, where the inputs that produce a desired motion are unknown and therefore must be computed. Specifically, maneuvering multibody dynamics (MMBD) (Bottasso, 2008) deals with the generation and execution of a plan for moving the virtual prototype of a vehicle from one location to another while achieving a given task. Typically, such plan must guarantee the satisfaction of certain constraints, for example due to the presence of obstacles, or necessary for ensuring that the vehicle remains within the boundaries of a finite performance envelope, or due to specific procedural requirements on the operation of the same vehicle. A typical application of MMBD is the analysis of maneuvering rotorcraft vehicles, for example in the study of the flare at the end of an auto-rotation, or of emergency maneuvers following the partial loss of power due to an engine failure during a take-off. For the latter case, figure 3 shows a series of snapshots from the maneuver computed for a detailed multibody model of a tilt-rotor using the procedures described in Bottasso, 2008.

A general framework for formulating complex maneuver problems at the boundaries of the operating envelope of a vehicle such as the one of figure 3, is provided by optimal control theory: a maneuver may be formulated as the solution which minimizes a cost function, for a given vehicle model as expressed by its governing equations of motion, and subject to all constraints on the system inputs and outputs which are necessary for fully characterizing the maneuver at hand. Indeed, optimal control theory offers a mathematically clear and constructive way for defining arbitrary maneuvers. However, with the ever increasing level of modeling detail and the complexity of current multibody vehicle models, especially in the case of multi-field models, the solution of optimal control problems is a daunting task. In fact, optimal control leads to the solution of two-point boundary value problems, rather than the classical initial value problems routinely solved by time-marching multibody codes, and boundary problems rapidly become of overwhelming computational cost as the complexity of the model grows. Bottasso, 2008 and references therein describe solution techniques which are applicable also to complex vehicle models.

A related, although somewhat simpler problem, is that of trim. For fixed wing aircrafts, steady trimmed flight simply means that control inputs are held fixed and the components of the vehicle linear and angular velocities are constant in a body attached frame. This is however not possible for a rotorcraft, which, being flown and controlled by means of rotating aerodynamic surfaces, is always excited by harmonic loads. Nonetheless, the vehicle controls and attitude can be set so as to achieve a particular periodic orbit; on this orbit, while the controls are constant in time, the system states are harmonic. Therefore a trimmed flight condition for a rotorcraft is characterized by having constant values of the average over a rotor revolution of the body components of the linear and angular velocities. Computing such a flight condition is a much more difficult problem than in the case of a fixed wing aircraft. The determination of the vehicle trim settings plays a crucial role in the analysis of the aeromechanic characteristics of a rotorcraft. Bottasso and Riviello, 2007 and references therein describe procedures for computing efficiently trimming control inputs by using suitable closed-loop controllers.

5. COROLLARY SUPPORTING TECHNOLOGIES

5.1. Parameter estimation

The objective of parameter estimation, a problem within the larger class of problems of system identification, is to find values of the parameters in a given mathematical model such that the model-computed response best matches (in a statistical sense) the experimentally observed one. More precisely, given a system $\mathcal{S}$ (the plant) and a suitable model of it $\mathcal{M}(\boldsymbol{p})$, parameterized in terms of free quantities $\boldsymbol{p}$, the problem of parameter estimation is concerned with finding values of the parameters $\boldsymbol{p}$ such that the model outputs $\boldsymbol{y}$ best match in some given sense some corresponding measured quantities $\boldsymbol{z}$, when both plant and model are excited by the same inputs $\boldsymbol{u}$. The problem is of a stochastic nature, since the plant is usually excited by a process noise $\widetilde{\boldsymbol{w}}$, while the observations are corrupted by a measurement noise $\widetilde{\boldsymbol{v}}$. This situation is illustrated in figure 4.

Clearly, effective parameter estimation techniques can have a profound and positive impact on multibody simulation technology, by closing the loop between virtual prototyping and testing of the actual hardware. Given the level of complexity of modern multibody models, which are multi-field and non-linear, the problem of parameter estimation is particularly challenging. Modern time domain methodologies (Jategaonkar, 2006) are the most promising ones in this respect. Two major classes of approaches are available, namely the batch optimization and the recursive filtering methods. Batch methods are one-shot optimization approaches that process all available data simultaneously to arrive at an estimate of the parameters. They are typically associated with a higher computational cost and are very strongly non-linear problems which may experience difficult convergence; however when they converge they typically provide rather reliable estimates. Recursive methods, on the other hand, process one sample data point at a time, and hence sweep rather swiftly through the data sets, to the point of often being applicable to real-time estimation problems for systems with time-varying parameters. The unknown parameters are however transformed into dynamic variables, and the relaxation towards steady state values is not always easy to achieve.

The application of such techniques to modern first-principle multibody vehicle models is still in its infancy, but we predict that its importance will grow substantially in the near future.

5.2. Model reduction

The response of the multibody system $\mathcal{M}$ to a time history of control inputs $\boldsymbol{u}(t)$ can be obtained by solving the governing equations (3) starting from given initial conditions. Accordingly, one may compute the response of the system outputs $\boldsymbol{y}(t)$.

Consider a second model $\hat{\mathcal{M}}$, described by some system of governing ODEs or DAEs expressed in terms of its own states $\hat{\boldsymbol{x}} \in \mathbb{R}^{\hat{n}}$. When subjected to the same time history of control inputs $\boldsymbol{u}(t)$, $\hat{\mathcal{M}}$ produces a time history of its own associated outputs $\hat{\boldsymbol{y}}(t)$.

Model $\hat{\mathcal{M}}$ is said to be a reduced model of $\mathcal{M}$ if $\hat{n} \ll n$ and

$$\hat{\boldsymbol{y}}(t) = \boldsymbol{y}(t) + \boldsymbol{e}(t), \tag{14}$$

where $\boldsymbol{e}(t)$ is the output error, small in an appropriate norm. Equation (14) states that, starting from the same given initial conditions, the reduced model $\hat{\mathcal{M}}$ and the original one $\mathcal{M}$ produce output responses which differ by the output error $\boldsymbol{e}(t)$ when subjected to the same input signal $\boldsymbol{u}(t)$. Clearly, the output error measures the fidelity of $\hat{\mathcal{M}}$ to $\mathcal{M}$. Typically, $\hat{\mathcal{M}}$ will give a good approximation of $\mathcal{M}$ at the slower scales, while the error $\boldsymbol{e}(t)$ will be mostly due to unmodeled or unresolved faster solution components.

The ability to reliably generate accurate reduced models from complex multibody systems can find applicability in several areas, as for example the synthesis of control laws, the evaluation of the stability of the system, real-time applications with high-frequency rates and/or limited computational resources, etc. (Brüls, Duysinx and Golinval, 2007; Bottasso, 2008).

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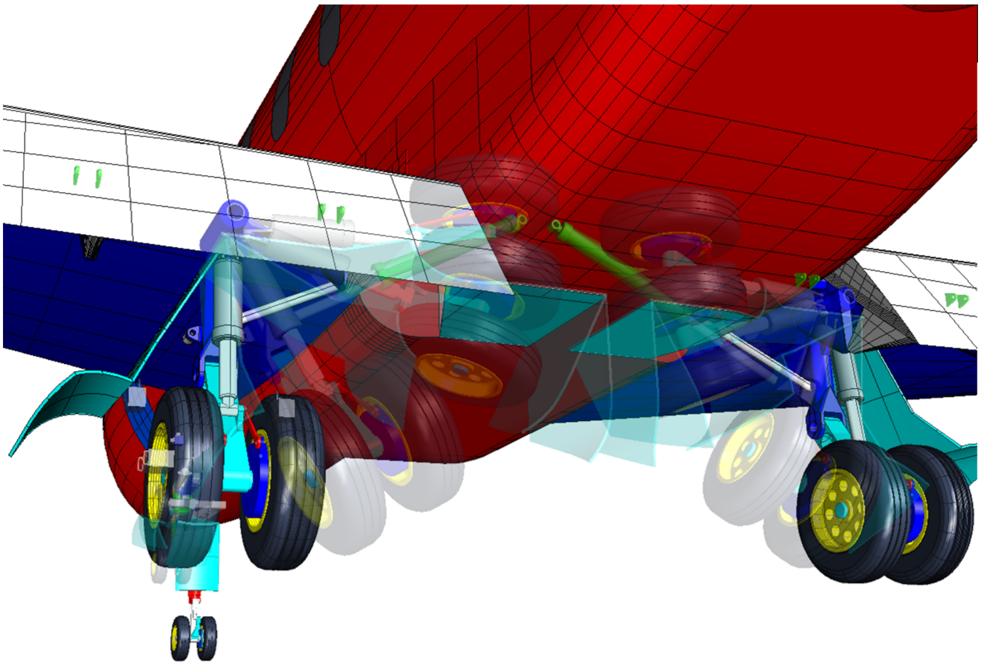

Figure 1. Multibody model of landing gear and flap systems (picture courtesy of MSC Software Corporation).

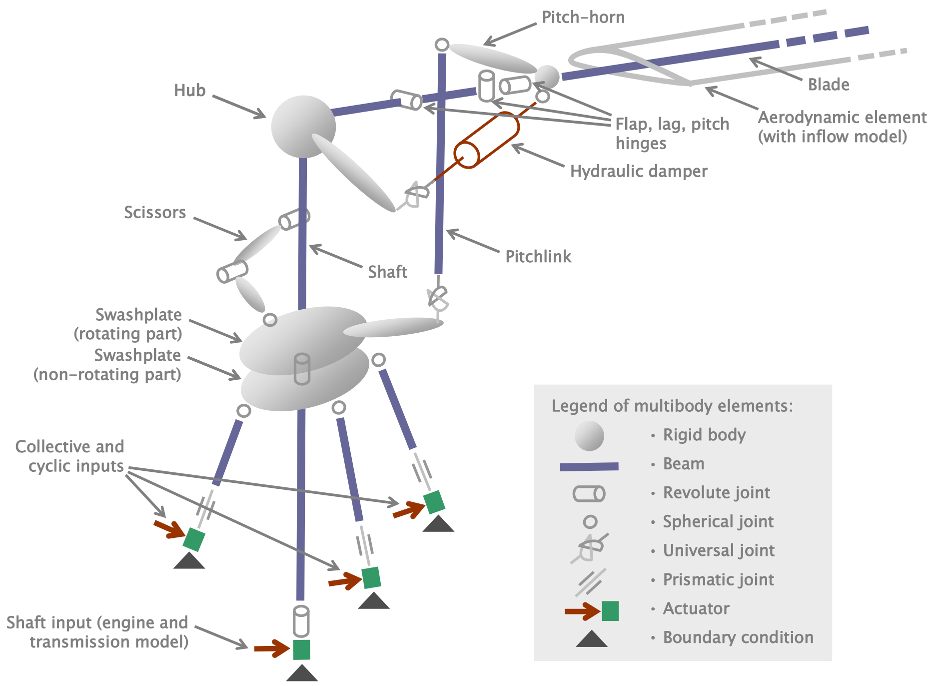

Figure 2. Topological view of the multibody model of a helicopter rotor with control linkages and lag damper (one single blade shown, for clarity).

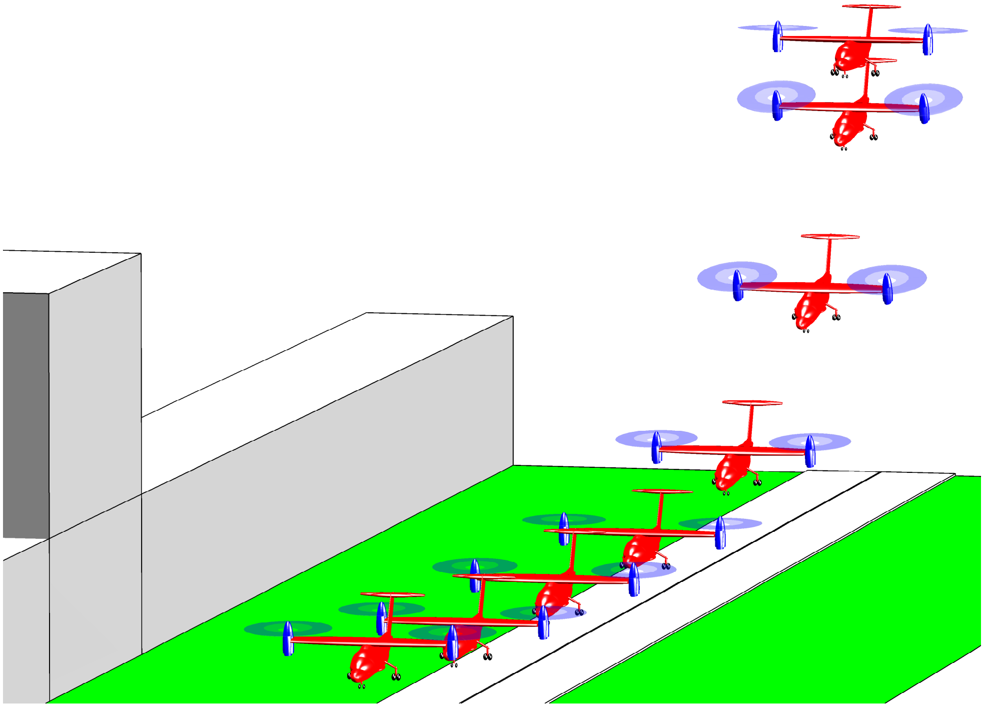

Figure 3. Continued take-off of a tilt-rotor, after loss of power from one of the engines.

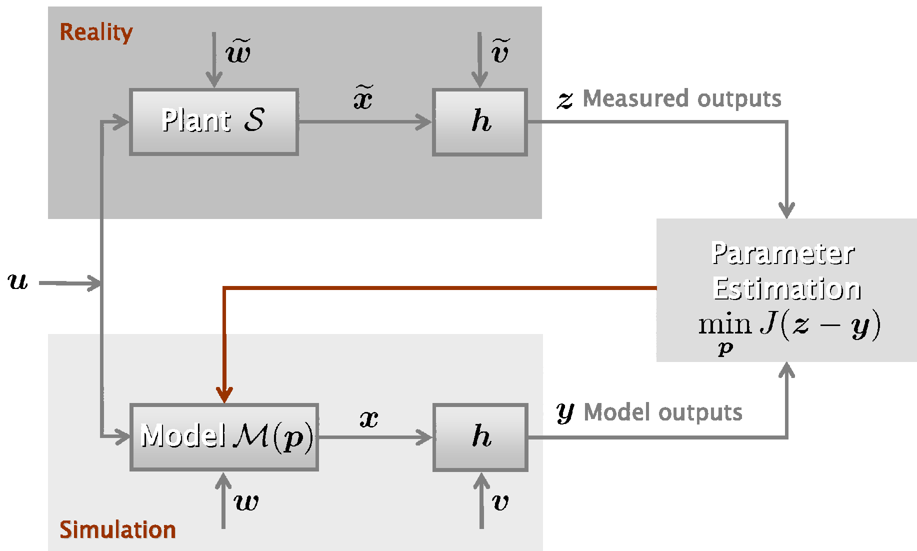

Figure 4. The problem of parameter estimation.