Traffic Flow Equations

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1 Introduction

In this report we examine a particular instance of a scalar conservation law in one dimension which is very interesting for the applications. Starting from the theory for a nonlinear hyperbolic equation, we study the equation modelling the traffic along a high-way without car inflow or outflow from external lanes. The analysis is based on the treatment of Randall LeVeque presented in Chapter 4 of Numerical Methods for Conservation Laws, Birkhäuser, Basel, 1992.

In Section 2, after the definition of the variable $\rho$ to represent the car density along the way together with its finite range of variation, the relationship between the car velocity $u$ and their density $\rho$ is introduced as function $u = u(\rho)$. Then the car flux $f$ is introduced by means of the definition $f(\rho) = \rho u(\rho)$. This traffic flux function is seen to be convex, namely $f'''(\rho) \leq 0$. Furthermore, the characteristic speed $a(\rho)$ of propagation of the information within the traffic flow will be derived. Finally, the propagation velocity $s$ of a density jump in the traffic flow is evaluated by means of the Rankine–Hugoniot condition. Of course, amongst the infinite weak solutions of any Cauchy problem associated with the nonlinear hyperbolic equation of the traffic flow, only one is physically acceptable and it is identified by satisfaction of the entropy condition. This condition will be expressed in the classical form of an inequality involving the value of $s$ and the characteristic velocities on the two sides of the discontinuity.

In Section 3 we will study the traffic flow problem for the linear model of the function $u(\rho)$. A sample of initial value problems with piecewise constant initial profiles will be solved. Some of them have continuous solution (rarefaction fans) while for some others the solution is discontinuous (propagating jumps).

Section 4 is devoted to case in which the velocity function $u(\rho)$ is logarithmic. Some of the initial value problems considered in Section 3 will be solved for this model of traffic flow.

2 Traffic equation: generality

Let us consider the flow of cars along a high-speed way, without inflow from and outflow to other ways. Let $\rho$ denote the density of cars, that is the number of vehicles per unit length. Of course, the value of the density variable $\rho$ must belong to a bounded interval

$$0 \leq \rho \leq \rho_{\text{jam}}$$

where $\rho_{\text{jam}}$ is the maximum possible car density with the cars bumper to bumper, as in the typical situation denoted as a traffic jam. Let us now indicate by $\hat{u}(x, t)$ the car velocity in a point $x$ of the highway at a certain time $t$. The total number of cars is conserved since the model excludes the presence of entrance and exit ways in the highway. As a consequence, the density and velocity must satisfy the conservation law for the number of cars, represented by the continuity equation in one dimension

$$\partial_t \rho + \partial_x (\rho u) = 0.$$  \hfill (2.1)
2.1 Conservation law in one dimension

As anticipated, in the considered traffic model the car velocity $u$ is a known function of the density $\rho$

$$u = u(\rho),$$

with the implicit understanding that $u \geq 0$ since driving backward or in reverse is not recommendable. A dependence of car velocity on their density is in accordance with the observations of real traffic flows: when we are on a highway we would like to drive optimally at a velocity $u_{\text{max}}$, maybe the existing speed limit, whereas in heavy traffic we are obliged to slow down. Therefore, the velocity $u$ decreases as the density $\rho$ increases with the function $u(\rho)$ in any case being defined only in the interval of the admitted values for the variable $\rho$, namely

$$u = u(\rho), \quad 0 \leq \rho \leq \rho_{j\text{am}}. \tag{2.2}$$

Therefore, we can introduce a flux function $f$, dependent only on $\rho$, which will be defined as follows

$$f(\rho) = \rho u(\rho), \quad 0 \leq \rho \leq \rho_{j\text{am}}. \tag{2.3}$$

Then, the conservation equation (2.1) assumes the form

$$\partial_t \rho + \partial_x f(\rho) = 0. \tag{2.4}$$

If both the function $f(\rho)$ and the solution $\rho(x, t)$ are differentiable, this equation can be written by developing the spatial derivative to give car conservation equation in quasilinear (advection) form

$$\partial_t \rho + a(\rho) \partial_x \rho = 0, \tag{2.5}$$

where

$$a(\rho) = f'(\rho) \tag{2.6}$$

represents the speed at which small perturbations propagate in the traffic flow of a given density $\rho$, a velocity called characteristic speed.

2.2 Speed of the discontinuity

For any weak solution of the considered hyperbolic equation it is possible to show that a jump between two different densities $\rho_l$ and $\rho_r$ propagates with a speed $s$ expressed by the well-known Rankine–Hugoniot jump relation

$$s = \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l}. \tag{2.7}$$

2.3 Entropy condition

For any initial value problem of a nonlinear hyperbolic equation there will be always an infinite number of weak solutions. The weak solutions comprise the
nonclassical, \textit{i.e.}, discontinuous, solutions, as well as the classical, \textit{i.e.}, continuous, ones. When the initial condition is continuous, a single continuous solution will exist but the solution can become, in some cases, discontinuous after a finite time, called \textit{breakdown time}. In any case, however, the weak solution is always nonunique and this is a common characteristic of all Cauchy problems associated with a nonlinear hyperbolic equation, irrespective of the continuous or discontinuous character of the initial data.

Amongst the infinite number of discontinuous weak solutions there is however only one that can be physically acceptable. This unique solution is selected by a condition that must be respected by any finite jump existing in the weak solution. More precisely, in order for a discontinuous solution to be acceptable from the physical viewpoint it is necessary that, for any jump between two values \( \rho_l \) and \( \rho_r \) in the solution, the following inequality

\[
f'(\rho_l) > s > f'(\rho_r),
\]

be satisfied, with the jump speed given by the Rankine–Hugoniot relation (2.7). This condition is called \textit{entropy condition}, although the word “entropy” here has no connection with the usual meaning of entropy in a thermodynamic context. This condition is valid for any convex flux, namely, for any flux function \( f(\rho) \) such that \( f''(\rho) \neq 0 \), for all \( \rho \) permitted: thus the condition applies equally to convex flux with the upward or downward concavity. On the contrary, its does not apply for a flux with a mixed upward and downward concavity. Thanks to the definition (2.6) of the characteristic speed, the entropy condition can be written also in the equivalent form

\[
a(\rho_l) > s > a(\rho_r).
\]

A weak solution satisfying condition (2.8) or (2.9) is said to be \textit{entropic} or to be the \textit{entropy solution}.

In particular, for a convex flux with a \textit{downward} concavity as in the case of the traffic equation under study (see below), it can be verified that the entropy condition is satisfied when the values \( \rho_l \) and \( \rho_r \) of the discontinuity in the solution are such that

\[
\rho_l < \rho_r.
\]

It is worthwhile to observe that this inequality is opposite to that pertaining to the inviscid Burgers equation, which reads instead as \( u_l > u_r \), and holds for a convex flux \( f(u) \) with upward concavity.

3 Model with linear velocity relation \( u = u(\rho) \)

3.1 Linear velocity relation

The simplest model to relate the car velocity with their density is the linear function

\[
u(\rho) = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{jam}}} \right), \quad 0 \leq \rho \leq \rho_{\text{jam}}.
\]
When the way is void ($\rho = 0$) car velocity will be equal to $u_{\text{max}}$ (Valentino drives bikes not cars), but decreases to zero as $\rho$ tends to $\rho_{\text{jam}}$. The flux function of the linear velocity model is then expressed by

$$f(\rho) = \rho u(\rho) = u_{\text{max}} \rho \left( 1 - \frac{\rho}{\rho_{\text{jam}}} \right), \quad 0 \leq \rho \leq \rho_{\text{jam}}. \quad (3.2)$$

By virtue of the preceding observations, the slope of the characteristic curves will be

$$a(\rho) = f'(\rho) = u_{\text{max}} \left( 1 - \frac{2\rho}{\rho_{\text{jam}}} \right), \quad 0 \leq \rho \leq \rho_{\text{jam}}. \quad (3.3)$$

The situation is depicted in the three plots of figure 3.1. Note that, although $u(\rho) \geq 0$, the characteristic speed $a(\rho)$ can be either positive or negative depending on the value of $\rho$ greater or less than $\frac{1}{2}\rho_{\text{jam}}$.

In the considered model, the speed of propagation of the discontinuities provided by Rankine–Hugoniot relation (2.7) is found to be

$$s = u_{\text{max}} \left( 1 - \frac{\rho_l + \rho_r}{\rho_{\text{jam}}} \right). \quad (3.4)$$

### 3.2 Dimensionless equation

It can be useful to write the conservation law also in dimensionless form. Let us start by defining dimensionless density

$$\tilde{\rho} = \frac{\rho}{\rho_{\text{jam}}}$$

By substituting this into the conservation law (2.4) we obtain

$$\partial_t (\rho_{\text{jam}} \tilde{\rho}) + \partial_x (u_{\text{max}} \rho_{\text{jam}} \tilde{\rho} (1 - \tilde{\rho})) = 0$$
Riminding that $\rho_{\text{jam}}$ and $u_{\text{max}}$ are constant, we have

$$\partial_t \tilde{\rho} + u_{\text{max}} \partial_x (\tilde{\rho} - \tilde{\rho}^2) = 0.$$  \hspace{1cm} (3.5)

Introducing now a reference length $L$ and a reference time $T$ defined as $T = \frac{L}{u_{\text{max}}}$, we can define the dimensionless length $\tilde{x}$ and time $\tilde{t}$

$$\begin{cases} \tilde{x} = x/L \\ \tilde{t} = t/T \end{cases}$$

Thanks to the change of variables, the partial derivatives will transform as follows

$$\partial_t = \partial_{\tilde{t}} \frac{d\tilde{t}}{dt} = \frac{u_{\text{max}}}{L} \partial_{\tilde{t}} \quad \text{and} \quad \partial_x = \partial_{\tilde{x}} \frac{d\tilde{x}}{dx} = \frac{1}{L} \partial_{\tilde{x}}$$

The dimensionless equation (3.5) of car conservation in advection form, after multiplying by $L/u_{\text{max}}$ will become

$$\partial_{\tilde{t}} \tilde{\rho} + \partial_{\tilde{x}} (\tilde{\rho} - \tilde{\rho}^2) = 0 \hspace{1cm} (3.6)$$

### 3.3 Bottle-neck problem

Let us now consider the traffic problem with the following initial condition

$$\rho(x, 0) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases}$$

in the specific case when

$$0 \leq \rho_l < \rho_r \leq \rho_{\text{jam}}$$

By virtue of the entropy condition (2.10), the propagating jump represents the entropic weak solution to the problem. We will study the solution behaviour for two different pairs of values $\rho_l$ and $\rho_r$. Note: in all graphs, it is always assumed that $u_{\text{max}} = 2$. 
**Example 3.1**

Consider the initial data:

\[ \rho(x, 0) = \begin{cases} \rho_l = \frac{1}{2} \rho_{jam} & x < 0 \\ \rho_r = \rho_{jam} & x > 0 \end{cases} \]

namely, the vehicles to the right of the origin are at rest and are caught by those coming from the left. Thus, their respective initial velocities are

\[ u_l = u(\rho_l) = u(\frac{1}{2} \rho_{jam}) = \frac{1}{2} u_{max} \quad \text{and} \quad u_r = u(\rho_r) = u(\rho_{jam}) = 0 \]

The flux, defined by the product of car velocity and density, is given by the linear relation (3.2). The slopes \( a(\rho) \) of the characteristics are:

\[ a_l = a(\rho_l) = a(\frac{1}{2} \rho_{jam}) = 0 \quad \text{and} \quad a_r = a(\rho_r) = a(\rho_{jam}) = -u_{max} \]

Since \( \rho_l = \frac{1}{2} \rho_{jam} < \rho_r = \rho_{jam} \), the entropy condition is satisfied and the initial discontinuity propagates to the left with the speed

\[ s = -\frac{1}{2} u_{max} < 0 \]

The density of the entropic solution consisting in the moving jump will be

\[ \rho(x, t) = \begin{cases} \frac{1}{2} \rho_{jam} & x < -\frac{1}{2} u_{max}t \\ \rho_{jam} & x > -\frac{1}{2} u_{max}t \end{cases} \]

The arriving vehicles must stop at the end of the queue of the cars at rest, which moves backward. The characteristics are shown in the second plot of figure 3.2.

![Figure 3.2: Left: initial condition. Right: characteristic lines of the example 3.1](image)

The trajectories (called, more properly, *universal curve*) can be calculated by solving the following ordinary differential initial-value problem

\[
\begin{cases}
\frac{dX}{dt} = \hat{u}(X, t) \\
X(0) = x_0
\end{cases}
\]

(3.7)
where \( \hat{u}(x, t) \) represents the velocity of the cars at position \( x \) and time \( t \), and \( x_0 \) is the initial point at \( t = 0 \) of the considered trajectory. The special letter \( \mathcal{X} \) has been used since the capital letter \( X \) is commonly employed to indicate the characteristic curves, which are generally different from the trajectories.

The velocity of the cars \( \hat{u}(x, t) \) can be obtained from the function \( u(\rho) \) of (3.1) of the considered velocity model, which depends only on variable \( \rho \), through the composition of functions:

\[
\hat{u}(x, t) = u(\rho(x, t)) \tag{3.8}
\]

By taking into account that the velocity values of the cars are uniform ahead and behind the jump, the car velocity at any point \( x \) and time \( t \) is given by

\[
\hat{u}(x, t) = \begin{cases} 
\frac{1}{2}u_{\text{max}} & x < -\frac{1}{2}u_{\text{max}}t \\
0 & x > -\frac{1}{2}u_{\text{max}}t 
\end{cases}
\]

The initial-value problem defining the trajectory assumes the form

\[
\begin{cases}
\frac{d\mathcal{X}}{dt} = u(\rho(\mathcal{X}, t)) \\
\mathcal{X}(0) = x_0
\end{cases}
\]

that, thanks to the solution \( \rho(x, t) \) just calculated, yields the differential equation

\[
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\frac{1}{2}u_{\text{max}} & \mathcal{X} < -\frac{1}{2}u_{\text{max}}t \\
0 & \mathcal{X} > -\frac{1}{2}u_{\text{max}}t 
\end{cases}
\]

The condition selecting either of the two possibilities is better recast in terms of the independent variable \( t \), so we have, actually,

\[
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\frac{1}{2}u_{\text{max}} & t < -2\mathcal{X}/u_{\text{max}} \\
0 & t > -2\mathcal{X}/u_{\text{max}} 
\end{cases}
\]

Unfortunately, the alternative cannot be resolved explicitly without examining the solution itself to the equation, \( \mathcal{X}(t) \). This implies that the alternative, once made explicit, will involve the value \( x_0 \) of the initial point of the trajectory. We consider therefore first a single trajectory with initial position to the left of the origin, namely such that \( \mathcal{X}(0) = x_0 < 0 \), and determine the precise form of the right-hand side of the equation for this particular situation. For \( x_0 < 0 \), the integration of the equation gives the trajectory

\[
\mathcal{X}(t) = x_0 + \frac{1}{2}u_{\text{max}}t.
\]

By substituting \( \mathcal{X}(t) \) into the condition \( t < -2\mathcal{X}/u_{\text{max}} \), it gives \( t < |x_0|/u_{\text{max}} \), so that the form of the differential equation with the alternative now made explicit will be, whenever \( x_0 < 0 \),

\[
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\frac{1}{2}u_{\text{max}} & t < |x_0|/u_{\text{max}} \\
0 & t > |x_0|/u_{\text{max}} 
\end{cases}
\]
The trajectories of the cars starting at any position $x_0$ can now be determined.

For the cars initially at $x_0 < 0$, one must distinguish two phases in the car motion: the car velocity is in a first phase equal to the constant value $\frac{1}{2}u_{\text{max}}$, so that the trajectory will be $X(t) = x_0 + \frac{1}{2}u_{\text{max}}t$. However, when the car crosses the backward moving jump it suddenly stops. This will occur at a time $t^* = |x_0|/u_{\text{max}}$. Therefore, the trajectories for the cars starting initially at $x_0 < 0$ are given by

$$X(t) = \begin{cases} x_0 + \frac{1}{2}u_{\text{max}}t & 0 \leq t \leq |x_0|/u_{\text{max}} \\ \frac{1}{2}x_0 & t \geq |x_0|/u_{\text{max}} \end{cases}$$

The cars initially at $x_0 > 0$ will be at rest forever and therefore for $x_0 > 0$ the trajectories will be, very simply, $X(t) = x_0$. The trajectories corresponding to the initial data of this example are shown in figure 3.3.

Figure 3.3: Car trajectories of the example 3.1
Figure 3.4: Left: initial condition. Right: characteristic lines of the example 3.2

**Example 3.2**

Let us consider now the following initial data

\[
\rho(x, 0) = \begin{cases} 
\rho_l = \frac{1}{4}\rho_{\text{jam}} & x < 0 \\
\rho_r = \frac{1}{2}\rho_{\text{jam}} & x > 0 
\end{cases}
\]

with the densities smaller than \(\rho_{\text{jam}}\) on both sides of the discontinuity. The initial velocities of the vehicles are

\[
u_l = u(\rho_l) = u\left(\frac{1}{4}\rho_{\text{jam}}\right) = \frac{3}{4}u_{\text{max}} \quad \text{and} \quad u_r = u(\rho_r) = u\left(\frac{1}{2}\rho_{\text{jam}}\right) = \frac{1}{2}u_{\text{max}}
\]

while the characteristic speeds on the two sides are

\[
a_l = a(\rho_l) = a\left(\frac{1}{4}\rho_{\text{jam}}\right) = \frac{1}{2}u_{\text{max}} \quad \text{and} \quad a_r = a(\rho_r) = a\left(\frac{1}{2}\rho_{\text{jam}}\right) = 0
\]

By means of Rankine–Hugoniot relation (3.4), the propagation speed of the considered discontinuity is found to be

\[s = \frac{1}{4}u_{\text{max}} > 0\]

In this case the jump moves to the right. The characteristic lines are drawn in the right plot of figure 3.4.

The car density \(\rho(x, t)\) of the solution for the given initial condition is

\[
\rho(x, t) = \begin{cases} 
\frac{1}{4}\rho_{\text{jam}} & x < \frac{1}{4}u_{\text{max}}t \\
\frac{1}{2}\rho_{\text{jam}} & x > \frac{1}{4}u_{\text{max}}t 
\end{cases}
\]

Using the solution \(\rho(x, t)\) just calculated and the relation \(\rho(x, t) = u(\rho(x, t))\), the differential equation for the trajectory reads

\[
\frac{dX}{dt} = \begin{cases} 
\frac{3}{4}u_{\text{max}} & X < \frac{1}{4}u_{\text{max}}t \\
\frac{1}{2}u_{\text{max}} & X > \frac{1}{4}u_{\text{max}}t
\end{cases}
\]

(3.9)
In terms of the time variable $t$ the equation for the trajectory reads

$$
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\frac{3}{4}u_{\text{max}} & t > 4\mathcal{X}/u_{\text{max}} \\
\frac{1}{2}u_{\text{max}} & t < 4\mathcal{X}/u_{\text{max}}
\end{cases} \quad (3.10)
$$

As in the previous example, the implicitness present in the alternative can be resolved only after including the initial condition of the trajectory, since the trajectories of the cars are different depending on their initial position $x_0 > 0$ or $x_0 < 0$.

Thus, consider the trajectory of a car starting on the left of the origin, from a point $x_0 < 0$. The equation of the trajectory will be

$$
\mathcal{X}(t) = x_0 + \frac{3}{4}u_{\text{max}}t. \quad (3.11)
$$

By substituting $\mathcal{X}(t)$ into the condition $t > 4\mathcal{X}/u_{\text{max}}$, it becomes $t < 2|x_0|/u_{\text{max}}$, so that the equation of the trajectory becomes, for $x_0 < 0$,

$$
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\frac{3}{4}u_{\text{max}} & t < 2|x_0|/u_{\text{max}} \\
\frac{1}{2}u_{\text{max}} & t > 2|x_0|/u_{\text{max}}
\end{cases} \quad (3.12)
$$

The trajectories of the cars initially in $x_0 < 0$ will consist in two parts since the car velocity will be constant and equal to $u_l = \frac{3}{4}u_{\text{max}}$, so that $\mathcal{X}(t) = x_0 + \frac{3}{4}u_{\text{max}}t$, but only up to the time $t^* = 2|x_0|/u_{\text{max}}$. For $t > 2|x_0|/u_{\text{max}}$, the car assumes the velocity $\frac{1}{2}u_{\text{max}}$ and the velocity change occurs when the car position is $x = \frac{1}{2}|x_0|$. In conclusion, the trajectories of the cars with initial position $x_0 < 0$ is

$$
\mathcal{X}(t) = \begin{cases} 
x_0 + \frac{3}{4}u_{\text{max}}t & 0 \leq t \leq 2|x_0|/u_{\text{max}} \\
-\frac{1}{2}|x_0| + \frac{1}{2}u_{\text{max}}t & t \geq 2|x_0|/u_{\text{max}}
\end{cases}
$$

The cars with initial position in $x_0 > 0$ will move with the constant velocity $u_r = \frac{1}{2}u_{\text{max}}$ and their trajectories will be, quite simply,

$$
\mathcal{X}(t) = x_0 + \frac{1}{2}u_{\text{max}}t \quad \forall t \geq 0.
$$

The trajectories so calculated are shown in the figure 3.5.
Figure 3.5: Car trajectories of the example 3.2
3.4 Rarefaction wave

When the entropy condition is not satisfied, the solution to the traffic flow equation must found by resorting to a method different from that employed in the previous examples. Let us suppose to look for a similar solution, that is a solution dependent on a single variable which is some combination of $x$ and $t$. To this purpose, let us note that the traffic equation

$$
\partial_t \rho + \partial_x f(\rho) = 0
$$

is invariant with respect to the transformation

$$
x \longrightarrow \lambda x, \quad t \longrightarrow \lambda t, \quad \forall \lambda > 0
$$

that is the solution does not change when the axes are scaled by the same parameter $\lambda$. Let us assume that the transformation is affine, namely, with $\lambda$ a constant. This suggest that the ratio $x/t$ is the suitable variable, and therefore we introduce the similarity variable

$$
\xi = \frac{x}{t}
$$

At the same time suppose that the solution is a function of $x$ and $t$ only through the new variable, namely we assume

$$
\rho(x, t) = \sigma \left( \frac{x}{t} \right) = \sigma(\xi)
$$

which means that the unknown is expressed as a single variable function indicated by $\sigma$. The new unknown variable $\sigma$ is called similarity unknown and must satisfy the ordinary differential equation

$$
\partial_t \sigma + \partial_x f(\sigma) = 0
$$

By developing the partial derivatives by means of the differentiation rule for composed functions (chain rule), one obtains

$$
\partial_t = -\frac{x}{t^2} \frac{d}{d\xi} \quad \text{and} \quad \partial_x = \frac{1}{t} \frac{d}{d\xi}
$$

and the equation for $\sigma$ becomes, after multiplying by $t$,

$$
\left[ -\frac{x}{t} + a(\sigma) \right] \sigma' = 0.
$$

Since we are in search for a nonconstant solution $\sigma' \neq 0$, the following algebraic relationship is obtained

$$
a(\sigma(\xi)) = \frac{x}{t} \quad \text{namely, equivalently,} \quad a(\sigma(\xi)) = \xi \quad (3.14)
$$

The expression (3.3) of $a(\sigma)$ for the linear velocity model gives

$$
a(\sigma(\xi)) = u_{\text{max}} \left( 1 - \frac{2\sigma(\xi)}{\rho_{\text{jam}}} \right) = \xi
$$
and by solving for $\sigma$:

$$\sigma(\xi) = a^{-1}(\xi) = \frac{\rho_{\text{jam}}}{2} \left( 1 - \frac{\xi}{u_{\text{max}}} \right), \quad |\xi| \le u_{\text{max}} \quad (3.15)$$

By summarizing, the original PDE has been reduced to an algebraic relation. It would seem that there is no place for taking into account the initial condition. Actually, the initial values can come into the play because they are necessary to define the interval of the inverse function $\sigma = a^{-1}(\xi)$ involved in the solution of the Riemann problem with considered initial condition (see figure 3.6).

![Figure 3.6: Function $\sigma = a^{-1}$ for the linear velocity model](image)

The function $\sigma(\xi)$ is defined in general for any value of $\xi$. However, in the considered problem $\sigma$ is comprised necessarily between the values $\rho_l$ and $\rho_r$, and thus the variable $\xi$ can vary only inside the interval $[\xi_l, \xi_r]$, where $\xi_l = a(\rho_l)$ and $\xi_r = a(\rho_r)$. The solution of the considered Riemann problem will assume the following form

$$\sigma(\xi) = \begin{cases} 
\rho_l & \xi > \xi_l \\
\frac{1}{2} \rho_{\text{jam}} \left( 1 - \frac{\xi}{u_{\text{max}}} \right) & \xi_l < \xi < \xi_r \\
\rho_r & \xi > \xi_r 
\end{cases} \quad (3.16)$$

namely, using the relation $\rho(x, t) = \sigma(x/t)$ in the interval $\xi_l \le \xi \le \xi_r$, to recover the solution for the original unknown $\rho$,

$$\rho(x, t) = \begin{cases} 
\rho_l & x < a(\rho_l) t \\
\frac{1}{2} \rho_{\text{jam}} \left( 1 - \frac{x}{u_{\text{max}} t} \right) & a(\rho_l) t < x < a(\rho_r) t \\
\rho_r & x > a(\rho_r) t 
\end{cases} \quad (3.17)$$
The car velocity \( \hat{u}(x, t) = u(\rho(x, t)) \) is found to be

\[
\hat{u}(x, t) = \begin{cases} 
  u_{\text{max}} \left(1 - \frac{\rho_l}{\rho_{\text{jam}}} \right) & x < a(\rho_l) t \\
  \frac{1}{2} u_{\text{max}} \left(1 + \frac{x}{u_{\text{max}} t} \right) & a(\rho_l) t < x < a(\rho_r) t \\
  u_{\text{max}} \left(1 - \frac{\rho_r}{\rho_{\text{jam}}} \right) & x > a(\rho_r) t
\end{cases}
\] (3.18)

A solution of this kind is called a \textit{rarefaction wave}, according to a term drawn from gasdynamics. Let us consider some example by assigning some specific values to the quantities \( \rho_l \) and \( \rho_r \), just to clarify the ideas.

### 3.5 Green semaphore problem

Let us now consider the following initial condition of the Riemann problem

\[
\rho(x, 0) = \begin{cases} 
  \rho_l & x < 0 \\
  \rho_r & x > 0
\end{cases}
\]

for the specific case in which

\[0 \leq \rho_r < \rho_l \leq \rho_{\text{jam}}\]

By virtue of the entropy condition (2.10) for the variable of the traffic equation, the propagation of this jump with the proper Rankine–Hugoniot speed corresponds to a nonentropic solution. Thus, the physically acceptable solution must be determined as a continuous solution. The classical solution will be sought for in the form of a similarity solution and will consists in a rarefaction fan.

Form the viewpoint of our traffic model, the considered initial condition represents the setting into motion of a queue of cars which are at rest on the left of semaphore, when the green light is turned on. The cars at the end of the queue remain at rest in a first moment but they begin to accelerate once the cars in front start moving. Since the velocity is connected with the density by the relation (3.1), any driver can augment its velocity only when the distance between himself and the preceding car increases. A gradual acceleration in the cars will be observed due to the decrease of the density.

Let us consider some example by assigning some specific values to the quantities \( \rho_l \) and \( \rho_r \), just to clarify the ideas.
Example 3.3

Consider the following initial condition for the Riemann problem

\[ \rho(x,0) = \begin{cases} \rho_{\text{jam}} & x < 0 \\ \frac{1}{2}\rho_{\text{jam}} & x > 0 \end{cases} \]

The left and right characteristic speeds are

\[ \xi_l = a(\rho_l) = a(\rho_{\text{jam}}) = -u_{\text{max}} \quad \text{and} \quad \xi_r = a(\rho_r) = a(\frac{1}{2}\rho_{\text{jam}}) = 0 \]

Therefore the density is

\[ \rho(x,t) = \begin{cases} \rho_{\text{jam}} & x < -u_{\text{max}} t \\ \frac{1}{2}\rho_{\text{jam}} \left( 1 - \frac{x}{u_{\text{max}} t} \right) & -u_{\text{max}} t < x < 0 \\ \frac{1}{2}\rho_{\text{jam}} & x > 0 \end{cases} \]

and the car velocity \( \hat{u}(x,t) = u(\rho(x,t)) \)

\[ \hat{u}(x,t) = \begin{cases} 0 & x < -u_{\text{max}} t \\ \frac{x}{2t} + \frac{u_{\text{max}}}{2} & -u_{\text{max}} t < x < 0 \\ \frac{1}{2}u_{\text{max}} & x > 0 \end{cases} \]

The differential equation governing the trajectories assumes the form

\[ \frac{d\mathcal{X}}{dt} = \begin{cases} 0 & \mathcal{X} < -u_{\text{max}} t \\ \frac{\mathcal{X}}{2t} + \frac{u_{\text{max}}}{2} & -u_{\text{max}} t < \mathcal{X} < 0 \\ \frac{1}{2}u_{\text{max}} & \mathcal{X} > 0 \end{cases} \]

The choice of one of the three possibilities has an implicit character since the inequalities involve the solution \( \mathcal{X}(t) \) itself. It is therefore necessary to consider a definite trajectory, which depends on the value of \( x_0 \), in order to make the selection explicit.

The cars starting at \( x_0 < 0 \) have trajectories consisting of two parts. These cars have \( \mathcal{X}(t) = x_0 < 0 \) so that the inequality \( \mathcal{X} < -u_{\text{max}} t \) becomes \( t < |x_0|/u_{\text{max}} \) and the cars remain at rest up to the time \( t = |x_0|/u_{\text{max}} \), which is different for each car. The equation of the trajectory assumes the form

\[ \frac{d\mathcal{X}}{dt} = \begin{cases} 0 & 0 < t < |x_0|/u_{\text{max}} \\ \frac{\mathcal{X}}{2t} + \frac{u_{\text{max}}}{2} & t > |x_0|/u_{\text{max}}, \mathcal{X} > 0 \end{cases} \]

At time \( t = |x_0|/u_{\text{max}} \), the rarefaction wave reaches the car and its velocity inside the fan must be used to compute the trajectory \( \mathcal{X}(t) \) of all cars inside the fan. This function is the solution of the differential equation

\[ \frac{d\mathcal{X}}{dt} = \frac{\mathcal{X}}{2t} + \frac{u_{\text{max}}}{2} \quad \text{(3.19)} \]
This is a nonhomogeneous linear, but with variable coefficients, differential equation of first order, whose solution is obtained by summing a particular solution to it with the general solution to the associated homogeneous equation. Let us suppose there exists a particular solution of the type 

$$X_p(t) = A \cdot t$$

and by substituting it into the equation (3.19) we obtain

$$A - \frac{A}{2} = \frac{u_{\text{max}}}{2} \quad \Rightarrow \quad A = u_{\text{max}}$$

Thus the function

$$X_p(t) = u_{\text{max}} \cdot t$$

is a particular solution to the complete equation. To find the general integral, we must determine the general solution of the homogeneous equation

$$\frac{dX}{dt} - \frac{X}{2t} = 0$$

which is an equidimensional (said also Euler) equation whose solution is of the type $X(t) = C \cdot t^\alpha$. By substituting this into the homogeneous equation we find

$$\alpha - \frac{1}{2} = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2}$$
so that the solution to the homogeneous equation is

\[ X_h(t) = C t^{\frac{1}{2}}. \]

This solution can be obtained also by the separation of variables. The general integral of the equation for the trajectory is therefore

\[ X(t) = C t^{\frac{1}{2}} + u_{\text{max}} t, \quad (3.20) \]

where \( C \) is the integration constant to be determined by imposing the initial condition for the fan.

The car that was at position \( x_0 < 0 \), initially, is affected by the expansion only after a time \( t^* \). At this time instant \( t = t^* \), when \( \rho_l = \rho_{\text{jam}} \), the machine is still in the initial position \( x_0 \). The first characteristic of the fan has a slope \(-u_{\text{max}}\), such that the value of time \( t^* \) satisfies

\[ -u_{\text{max}} = \frac{x_0}{t^*} \quad \text{namely} \quad t^* = \frac{|x_0|}{u_{\text{max}}}. \]

By imposing the condition at the (new) "initial" time \( t = t^* \), the relation (3.20) provides

\[ x_0 = C \sqrt{\frac{|x_0|}{u_{\text{max}}}} + u_{\text{max}} \frac{|x_0|}{u_{\text{max}}}. \]

By solving for \( C \) we obtain

\[ C = -2\sqrt{|x_0|} u_{\text{max}}. \]

Thus the trajectory of cars inside the fan is

\[ X(t) = -2\sqrt{|x_0|} u_{\text{max}} t + u_{\text{max}} t. \]

The last characteristic of the fan is the straight line \( x = 0 \), and therefore to find the instant \( t_{\text{fin}} \) when the car goes out from the fan the equation

\[ X(t_{\text{fin}}) = 0 \]

must be solved, namely:

\[ -2\sqrt{|x_0|} u_{\text{max}} t_{\text{fin}} + u_{\text{max}} t_{\text{fin}} = 0, \]

from which one obtains

\[ t_{\text{fin}} = \frac{4|x_0|}{u_{\text{max}}}. \]

Therefore, by summarizing, the trajectory of a car initially in \( x_0 < 0 \) is represented by the function

\[ X(t) = \begin{cases} x_0 & t \leq \frac{|x_0|}{u_{\text{max}}} \\ -2\sqrt{|x_0|} u_{\text{max}} t + u_{\text{max}} t & \frac{|x_0|}{u_{\text{max}}} \leq t \leq \frac{4|x_0|}{u_{\text{max}}} \\ -2|x_0| + \frac{1}{2} u_{\text{max}} t & t \geq \frac{4|x_0|}{u_{\text{max}}} \end{cases} \]

The cars initially at \( x_0 > 0 \) have always the constant velocity \( u_r = u(\rho_r) = u(\frac{1}{2} \rho_{\text{jam}}) = \frac{1}{2} u_{\text{max}} \) and their trajectories are

\[ X(t) = x_0 + \frac{1}{2} u_{\text{max}} t, \quad \forall t \geq 0. \quad (3.21) \]

In figure 3.8 the trajectories of all cars are depicted.
Figure 3.8: Car trajectories of the example 3.3
Example 3.4

Let us now suppose that the initial condition of the Riemann problem is

$$\rho(x, 0) = \begin{cases} 
\rho_{\text{jam}} & x < 0 \\
0 & x > 0
\end{cases}$$

The left and right characteristic speeds are

$$\xi_l = a(\rho_l) = a(\rho_{\text{jam}}) = -u_{\text{max}} \quad \text{and} \quad \xi_r = a(\rho_r) = a(0) = u_{\text{max}}$$

Using the function $\sigma(x/t)$ in the rarefaction fan, the density $\rho(x, t)$ is found to be

$$\rho(x, t) = \begin{cases} 
\rho_{\text{jam}} & x < -u_{\text{max}} t \\
\frac{1}{2} \rho_{\text{jam}} \left(1 - \frac{x}{u_{\text{max}} t}\right) & -u_{\text{max}} t < x < u_{\text{max}} t \\
0 & x > u_{\text{max}} t
\end{cases}$$

By relation (3.1) of the linear velocity model, the car velocity $\dot{u}(x, t) = u(\rho(x, t))$ is

$$\dot{u}(x, t) = \begin{cases} 
0 & x < -u_{\text{max}} t \\
\frac{x}{2t} + \frac{u_{\text{max}}}{2} & -u_{\text{max}} t < x < u_{\text{max}} t \\
u_{\text{max}} & x > u_{\text{max}} t
\end{cases}$$

The second plot in figure 3.9 describes the characteristic lines of the solution. The equation for the trajectories is

$$\frac{dX}{dt} = \begin{cases} 
0 & X < -u_{\text{max}} t \\
\frac{X}{2t} + \frac{u_{\text{max}}}{2} & -u_{\text{max}} t < X < u_{\text{max}} t \\
u_{\text{max}} & X > u_{\text{max}} t
\end{cases}$$

The three conditions that define the right-hand side can be made explicit in terms of the independent variable $t$ by considering a specific trajectory, which is different for each initial position $x_0$, either positive or negative. For $x_0 < 0$, the cars remain at rest and their trajectory is $X(t) = x_0$ up to the time $t^*$ when the rarefaction wave, left propagating at a speed $-u_{\text{max}}$, reaches them. This time is provided by the condition $x_0 = -u_{\text{max}} t^*$, which gives $t^* = -x_0/u_{\text{max}} = |x_0|/u_{\text{max}}$. The equation of the trajectory becomes

$$\frac{dX}{dt} = \begin{cases} 
0 & 0 < t < |x_0|/u_{\text{max}} \\
\frac{X}{2t} + \frac{u_{\text{max}}}{2} & t > |x_0|/u_{\text{max}}, X > u_{\text{max}} t \\
u_{\text{max}} & X > u_{\text{max}} t
\end{cases}$$

Inside the fan the trajectory is provided by the solution $X(t) = C\sqrt{t} + u_{\text{max}} t$, with the constant $C$ to be determined by imposing the “initial condition” at time $t =
Figure 3.9: Left: solution at $t > 0$. Right: characteristic lines and fan of the example 3.4

$|x_0|/u_{\text{max}}$: $\mathcal{X}(|x_0|/u_{\text{max}}) = x_0$. We have the equation $C \sqrt{|x_0|/u_{\text{max}}} + |x_0| = x_0$, whose solution is $C = -2 \sqrt{|x_0|/u_{\text{max}}}$. Thus, the part of the trajectory inside the expansion wave is

$$\mathcal{X}(t) = -2 \sqrt{|x_0|/u_{\text{max}}} t + u_{\text{max}} t$$

The condition $\mathcal{X} > u_{\text{max}} t$ is seen to be always satisfied which means that the cars remain inside the rarefaction wave forever. By combining the two pieces of the trajectory, its complete expression is found in the form

$$\mathcal{X}(t) = \begin{cases} x_0 & 0 \leq t \leq |x_0|/u_{\text{max}} \\ -2 \sqrt{|x_0|/u_{\text{max}}} t + u_{\text{max}} t & t \geq |x_0|/u_{\text{max}} \end{cases}$$

For $x = x_0 > 0$, the cars never change their initial velocity $u_{\text{max}}$ and therefore their trajectories are simply

$$\mathcal{X}(t) = x_0 + u_{\text{max}} t, \quad \forall t \geq 0.$$
Figure 3.10: Car trajectories of the example 3.4
Example 3.5

The initial condition is now:

\[
\rho(x, 0) = \begin{cases} 
\frac{3}{4} \rho_{\text{jam}} & x < 0 \\
\frac{1}{4} \rho_{\text{jam}} & x > 0 
\end{cases}
\]

From relation \( \xi = a(\rho) \) we obtain immediately

\[
\rho_l = \frac{3}{4} \rho_{\text{jam}} \quad \rightarrow \quad \xi_l = a_l = a(\frac{3}{4} \rho_{\text{jam}}) = -\frac{1}{2} u_{\text{max}}
\]

and in the same manner

\[
\rho_r = \frac{1}{4} \rho_{\text{jam}} \quad \rightarrow \quad \xi_r = a_r = a(\frac{1}{4} \rho_{\text{jam}}) = \frac{1}{2} u_{\text{max}}
\]

The density becomes

\[
\rho(x, t) = \begin{cases} 
\frac{3}{4} \rho_{\text{jam}} & x < -\frac{1}{2} u_{\text{max}} t \\
\frac{1}{2} \rho_{\text{jam}} \left(1 - \frac{x}{u_{\text{max}} t}\right) & -\frac{1}{2} u_{\text{max}} t < x < \frac{1}{2} u_{\text{max}} t \\
\frac{1}{4} \rho_{\text{jam}} & x > \frac{1}{2} u_{\text{max}} t
\end{cases}
\]

For the linear velocity model the car velocity \( \hat{u}(x, t) = u(\rho(x, t)) \) is found to be given by the function

\[
\hat{u}(x, t) = \begin{cases} 
\frac{1}{4} u_{\text{max}} & x < -\frac{1}{2} u_{\text{max}} t \\
\frac{x}{2t} + \frac{u_{\text{max}}}{2} & -\frac{1}{2} u_{\text{max}} t < x < \frac{1}{2} u_{\text{max}} t \\
\frac{3}{4} u_{\text{max}} & x > \frac{1}{2} u_{\text{max}} t
\end{cases}
\]

The equation governing the trajectories is

\[
\frac{dX}{dt} = \begin{cases} 
\frac{1}{4} u_{\text{max}} & X < -\frac{1}{2} u_{\text{max}} t \\
\frac{X}{2t} + \frac{u_{\text{max}}}{2} & -\frac{1}{2} u_{\text{max}} t < X < \frac{1}{2} u_{\text{max}} t \\
\frac{3}{4} u_{\text{max}} & X > \frac{1}{2} u_{\text{max}} t
\end{cases}
\]

As in the previous problem, the trajectories of the cars depending on their initial position: \( x_0 > 0 \) or \( x_0 < 0 \).

The cars initially at \( x_0 < 0 \) have the constant velocity \( u_l = \frac{1}{4} u_{\text{max}} \) so their trajectory is

\[
X(t) = x_0 + \frac{1}{4} u_{\text{max}} t,
\]

up to a time \( t^* \) when the car reaches the first ray of the rarefaction wave. The latter is propagating to the left with speed \( -\frac{1}{2} u_{\text{max}} \). The value of \( t^* \) is found from the equation \( x_0 + \frac{1}{4} u_{\text{max}} t^* = -\frac{1}{2} u_{\text{max}} t^* \), which gives \( t^* = \frac{4}{3} |x_0| / u_{\text{max}} \). Therefore the equation for the trajectory assumes the form

\[
\frac{dX}{dt} = \begin{cases} 
\frac{1}{4} u_{\text{max}} & 0 < t < \frac{4}{3} |x_0| / u_{\text{max}} \\
\frac{X}{2t} + \frac{u_{\text{max}}}{2} & t > \frac{4}{3} |x_0| / u_{\text{max}}, \ X < \frac{1}{2} u_{\text{max}} t \\
\frac{3}{4} u_{\text{max}} & X > \frac{1}{2} u_{\text{max}} t
\end{cases}
\]
Since time \( t = \frac{4}{3}|x_0|/u_{\text{max}} \), which is different for each car, the velocity of the car is that of the rarefaction wave and therefore the trajectory inside the fan is given by the solution
\[
X_f(t) = C \sqrt{t} + u_{\text{max}} t,
\]
with the constant \( C \) determined by the “initial” condition: \( X_f(\frac{4}{3}|x_0|/u_{\text{max}}) = X(\frac{4}{3}|x_0|/u_{\text{max}}) \), giving the equation
\[
C \sqrt{\frac{4}{3}|x_0|/u_{\text{max}}} + \frac{4}{3}|x_0| = x_0 + \frac{1}{3}|x_0|
\]
which yields \( C = -\sqrt{3}|x_0|u_{\text{max}} \). Thus, the part of the trajectory of the cars passing through the rarefaction wave is
\[
X_f(t) = -\sqrt{3}|x_0|u_{\text{max}} t + u_{\text{max}} t
\]
For \( t \) large enough the car velocity \( \rightarrow u_{\text{max}} \) but this value is higher than the value \( \frac{3}{4}u_{\text{max}} \) of all cars ahead of the rarefaction. This means that the cars inside the rarefaction wave reach its right limit, which is right propagating at the speed \( \frac{1}{2}u_{\text{max}} \). This occurs at a time \( t^* \), which depends of course on the car, and can be obtained by the condition \( X_f(t^*) = \frac{1}{2}u_{\text{max}} t^* \). Using the expression of \( X_f(t) \) just calculated, we have the equation
\[
-\sqrt{3}|x_0|u_{\text{max}} t^* + u_{\text{max}} t^* = \frac{1}{2}u_{\text{max}} t^*,
\]
whose solution is found to be \( t^* = 12|x_0|/u_{\text{max}} \). From time \( t = 12|x_0|/u_{\text{max}} \) on the trajectory becomes a straight line of slope \( \frac{3}{4}u_{\text{max}} \), thus
\[
X(t) = K + \frac{3}{2}u_{\text{max}} t,
\]
with constant \( K \) to be determined by imposing the continuity of the trajectory at time \( t^* = 12|x_0|/u_{\text{max}} \), namely by the condition \( X_f(12|x_0|/u_{\text{max}}) = K + 9u_{\text{max}} \), which gives \( K = 3x_0 = -3|x_0| \).

The complete expression of the trajectories of cars with initial position \( x = x_0 < 0 \) is found to be:
\[
X(t) = \begin{cases} 
  x_0 + \frac{1}{4}u_{\text{max}} t & 0 \leq t \leq \frac{4}{3}|x_0|/u_{\text{max}} \\
  -\sqrt{3}|x_0|u_{\text{max}} t + u_{\text{max}} t & \frac{4}{3}|x_0|/u_{\text{max}} \leq t \leq 12|x_0|/u_{\text{max}} \\
  3x_0 + \frac{3}{2}u_{\text{max}} t & t \geq 12|x_0|/u_{\text{max}}
\end{cases}
\]

For cars with initial position \( x_0 > 0 \) the velocity remains constant equal to \( \frac{3}{4}u_{\text{max}} \) so that the trajectory is given by the simple equation
\[
X(t) = x_0 + \frac{3}{4}u_{\text{max}} t, \quad \forall t \geq 0.
\]

Figure 3.11 contains the trajectories of all cars.
Figure 3.11: Car trajectories of the example 3.5
4 Logarithmic model of the relation \( u = u(\rho) \) (Greenberg)

4.1 Greenberg velocity relation

This model has been used for the study and design of the Lincoln tunnel and was proposed by Greenberg in 1959. In this model the linear velocity relation (3.1) is replaced by the following logarithmic function

\[
    u(\rho) = \alpha \ln \left( \frac{\rho_{\text{jam}}}{\rho} \right), \quad 0 \leq \rho \leq \rho_{\text{jam}} \tag{4.1}
\]

Actually, the coefficient \( \alpha \) (which dimensionally is a velocity) is not a constant but depends on the position and on time

\[
    \alpha = \alpha(x, t)
\]

although in the present treatment is considered a constant. The car flux is always defined by the product of the density and the velocity of the cars, namely,

\[
    f(\rho) = \alpha \rho \ln \left( \frac{\rho_{\text{jam}}}{\rho} \right), \quad 0 \leq \rho \leq \rho_{\text{jam}}
\]

and the characteristic speed is its derivative

\[
    a(\rho) = f'(\rho) = \alpha \left[ \ln \left( \frac{\rho_{\text{jam}}}{\rho} \right) - 1 \right], \quad 0 \leq \rho \leq \rho_{\text{jam}}
\]

Figure 4.1: (a) logarithmic velocity relation, (b) flux, (c) characteristic velocity of the Greenberg traffic model
4.2 Dimensionless equation

Also in the model based on the logarithmic relation it is possible to derive a dimensionless counterpart of the traffic equation. However, in this case it is convenient to adimensionalize the density, the space and time by means of quantities different from those considered for the linear model. The previous adimensionalization made it possible to eliminate \( u_{\text{max}} \) from the equation, while the advantage of the present adimensionalization is that it eliminates the Greenberg constant \( \alpha \).

The dimensionless density is defined by

\[
\tilde{\rho} = \frac{\rho}{\rho_{\text{jam}}}
\]

and is substituted into the traffic equation to yield

\[
\partial_t \tilde{\rho} + \partial_x \left[ \alpha \tilde{\rho} \ln \left( \frac{1}{\tilde{\rho}} \right) \right] = 0
\]

Now it is necessary to make dimensionless the coordinates of the kinematic plane; the dimensionless variables allowing to simplify \( a \) are

\[
\tilde{x} = \frac{x}{L} \quad \text{and} \quad \tilde{t} = \frac{\alpha}{L} t
\]

where the constant \( \alpha \) has been used as the reference velocity. By means of simple algebraic calculations and by exploiting \( \alpha = \text{constant} \), we obtain the dimensionless form of traffic equation

\[
\partial_t \tilde{\rho} + \partial_x \left[ \tilde{\rho} \ln \left( \frac{1}{\tilde{\rho}} \right) \right] = 0
\] (4.2)

4.3 Bottle-neck problem

Let us consider the initial condition for the Riemann problem

\[
\rho(x, 0) = \begin{cases} 
\rho_l & x < 0 \\
\rho_r & x > 0 
\end{cases}
\]

under the limitation

\[
0 \leq \rho_l < \rho_r \leq \rho_{\text{jam}}
\]

Let us now examine the same two particular examples that have been considered with the linear velocity relation and let us determine the solution for the logarithmic \( u(\rho) \).
Example 4.1

Consider the following specific initial condition

\[ \rho(x, 0) = \begin{cases} \rho_l = \frac{1}{2} \rho_{\text{jam}} & x < 0 \\ \rho_r = \rho_{\text{jam}} & x > 0 \end{cases} \]

The car velocity in these states is obtained from the relation (4.1) of the Greenberg model:

\[ u(\rho_l) = u(\frac{1}{2} \rho_{\text{jam}}) = \alpha \ln 2 \quad \text{and} \quad u(\rho_r) = u(\rho_{\text{jam}}) = \alpha \ln 1 = 0 \]

The characteristic speeds are:

\[ a_l = a(\rho_l) = a(\frac{1}{2} \rho_{\text{jam}}) = (\ln 2 - 1) \alpha \quad \text{and} \quad a_r = a(\rho_r) = a(\rho_{\text{jam}}) = -\alpha \]

One can deduce that the jump moves toward the left with speed

\[ s = \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l} = -\alpha \ln 2 < 0 \]

The car density of the solution is

\[ \rho(x, t) = \begin{cases} \frac{1}{2} \rho_{\text{jam}} & x < -(\ln 2) \alpha t \\ \rho_{\text{jam}} & x > -(\ln 2) \alpha t \end{cases} \]

Therefore the distribution of car velocities is

\[ \hat{u}(x, t) = \begin{cases} \alpha \ln 2 & x < -(\ln 2) \alpha t \\ 0 & x > -(\ln 2) \alpha t \end{cases} \]
The equation governing the trajectories of the cars is

\[
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\alpha \ln 2 & \mathcal{X} < -(\ln 2) t^* \\
0 & \mathcal{X} > -(\ln 2) t^* 
\end{cases}
\]

The trajectories of the cars depend on the positive or negative value of \(x_0\).

For cars initially at \(x_0 < 0\), the trajectories consist of two parts and the cars move to the right with velocity \(u_{\text{max}}\). Their trajectory is initially

\[
\mathcal{X}(t) = x_0 + u_{\text{max}} t.
\]

Then, the car reaches the end of the queue at time \(t^*\) defined by \(x_0 + u_{\text{max}} t^* = -(\ln 2)\alpha t^*\) which gives \(t^* = |x_0|/(u_{\text{max}} + \alpha \ln 2)\).

For larger times, the cars stop and remain in a fixed position. Therefore the complete expression of the trajectory of cars with \(x_0 < 0\) will be

\[
\mathcal{X}(t) = \begin{cases} 
x_0 + u_{\text{max}} t & 0 \leq t \leq \frac{|x_0|}{u_{\text{max}} + \alpha \ln 2} \\
\frac{|x_0|}{1 + u_{\text{max}}/(\alpha \ln 2)} & t \geq \frac{|x_0|}{u_{\text{max}} + \alpha \ln 2}
\end{cases}
\]

For cars initially at \(x_0 > 0\), they remain at rest, so their trajectory is

\[
\mathcal{X}(t) = x_0, \quad \forall t \geq 0.
\]
Example 4.2

Consider now the following initial data:

\[ \rho(x, 0) = \begin{cases} \rho_l = \frac{1}{4} \rho_{\text{jam}} & x < 0 \\ \rho_r = \frac{1}{2} \rho_{\text{jam}} & x > 0 \end{cases} \]

The velocities are

\[ u(\rho_l) = u\left(\frac{1}{4} \rho_{\text{jam}}\right) = \alpha \ln 4 \quad \text{and} \quad u(\rho_r) = u\left(\frac{1}{2} \rho_{\text{jam}}\right) = \alpha \ln 2 = \frac{1}{2} u(\rho_l) \]

while the Greenberg flux, once taken the derivative, provides the characteristic speeds

\[ a(\rho_l) = a\left(\frac{1}{4} \rho_{\text{jam}}\right) = (\ln 4 - 1) \alpha \quad \text{and} \quad a(\rho_r) = a\left(\frac{1}{2} \rho_{\text{jam}}\right) = (\ln 2 - 1) \alpha \]

By means of the Rankine–Hugoniot jump condition the propagation speed of the discontinuity is immediately found

\[ s = \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l} = -4 \alpha \left(\frac{1}{2} \ln 2 - \frac{1}{4} \ln 4\right) = 0 \]

In this particular case the discontinuity is stationary.

Figure 4.4: Left: initial condition. Right: characteristic lines of the example 4.2

Since the discontinuity is stationary, the initial density remain for any \( t > 0 \). The car density of the solution is therefore

\[ \rho(x, t) = \begin{cases} \frac{1}{4} \rho_{\text{jam}} & x < 0 \\ \frac{1}{2} \rho_{\text{jam}} & x > 0 \end{cases} \]

Therefore the distribution of car velocities is

\[ \hat{u}(x, t) = \begin{cases} \alpha \ln 4 & x < 0 \\ \alpha \ln 2 & x > 0 \end{cases} \]
The equation governing the trajectories of the cars is

\[
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\alpha \ln 4 & \mathcal{X} < 0 \\
\alpha \ln 2 & \mathcal{X} > 0 
\end{cases}
\]

The trajectories of the cars depend on the positive or negative value of \( x_0 \).

For cars initially at \( x_0 < 0 \), the trajectories consist of two parts since the cars move to right with velocity \( (\ln 4)\alpha \). Their trajectory is initially

\[
\mathcal{X}(t) = x_0 + (\ln 4)\alpha t
\]

Then the cars reach the end of the queue at time \( t^* \) defined by

\[
x_0 + (\ln 4)\alpha t^* = 0
\]

which gives \( t^* = |x_0|/(\alpha \ln 4) \). For larger times, the cars stop and remain in a fixed position. Therefore, the complete expression of the trajectory of cars with \( x_0 < 0 \) will be

\[
\mathcal{X}(t) = \begin{cases} 
x_0 + (\ln 4)\alpha t & 0 \leq t \leq \frac{|x_0|}{(\ln 4)\alpha} \\
(\ln 2) \left( \frac{|x_0|}{\ln 4} + \alpha t \right) & t \geq \frac{|x_0|}{(\ln 4)\alpha}
\end{cases}
\]

For cars initially at \( x_0 > 0 \), their velocity is \( (\ln 2)\alpha \) so that the expression of the trajectory is

\[
\mathcal{X}(t) = x_0 + (\ln 2)\alpha t, \quad \forall t \geq 0.
\]
Figure 4.5: Car trajectories of the example 4.2
4.4 Rarefaction wave

Let us consider a Riemann problem with initial condition

\[
\rho(x, 0) = \begin{cases} 
\rho_l & x < 0 \\
\rho_r & x > 0 
\end{cases}
\]

where \(0 \leq \rho_r < \rho_l \leq \rho_{\text{jam}}\)

As in the linear model, the solution will be continuous and the transition from the value \(\rho_l\) to the value \(\rho_r\) will occur through an expansion fan. Also in this case we solve the problem looking for a similarity solution \(\sigma(\xi)\) that is a function of the single similarity variable \(\xi = x/t\). By introducing the similar solution in the traffic equation, we arrive at the following relation

\[
\sigma(\xi) = \rho_{\text{jam}} e^{-(\xi + 1)/\alpha} \tag{4.3}
\]

where \(\alpha\) still denotes the value of the Greenberg constant. The solution of the initial value problem above is

\[
\rho(x, t) = \begin{cases} 
\rho_l & x < \xi_l t \\
\rho_{\text{jam}} e^{-(\xi_l + 1)/\alpha} & \xi_l t < x < \xi_r t \\
\rho_r & x > \xi_r t 
\end{cases}
\]

where

\[
\xi_l = a_l = a(\rho_l) = \alpha \ln \left( \frac{\rho_{\text{jam}}}{\rho_l} \right) - 1
\]
and
\[ \xi_r = a_r = a(\rho_r) = \alpha \left[ \ln \left( \frac{\rho_{jam}}{\rho_r} \right) - 1 \right] \]

The car velocity \( \hat{u}(x, t) = u(\rho(x, t)) \) is

\[
\hat{u}(x, t) = \begin{cases} 
\alpha \ln \left( \frac{\rho_{jam}}{\rho_l} \right) & x < \xi t \\
\alpha + \frac{x}{t} & \xi t < x < \xi_r t \\
\alpha \ln \left( \frac{\rho_{jam}}{\rho_r} \right) & x > \xi_r t
\end{cases}
\]

### 4.5 Trajectories inside the rarefaction wave

The trajectory of a car is the solution \( \mathcal{X}(t) \) to the differential equation

\[
\frac{d\mathcal{X}}{dt} = \hat{u}(\mathcal{X}, t) = u(\rho(\mathcal{X}, t))
\]

Using the car velocity just calculated, the equation for the trajectory becomes

\[
\frac{d\mathcal{X}}{dt} = \begin{cases} 
\alpha \ln \left( \frac{\rho_{jam}}{\rho_l} \right) & \mathcal{X} < \xi t \\
\alpha + \frac{\mathcal{X}}{t} & \xi t < \mathcal{X} < \xi_r t \\
\alpha \ln \left( \frac{\rho_{jam}}{\rho_r} \right) & \mathcal{X} > \xi_r t
\end{cases}
\]

where \( \mathcal{X}(t) \) denotes the instantaneous position of a car.

Let us now determine the trajectories of cars that at the initial time \( t = 0 \) occupy any position \( x_0 \).

Outside the expansion fan the velocity is constant and thus the trajectory in the space-time plane is a straight line

\[
\mathcal{X}(t) = x_0 + \left[ \ln \left( \frac{\rho_{jam}}{\rho_l} \right) \right] \alpha t, \quad \text{for } 0 \leq t \leq \frac{|x_0|}{\alpha}
\]

since at time \( t = \frac{|x_0|}{\alpha} \) the car starts to accelerate. On the contrary, inside the expansion fan the trajectory is found by solving the ODE

\[
\frac{d\mathcal{X}}{dt} - \frac{\mathcal{X}}{t} = \alpha
\]

for \( t \geq \frac{|x_0|}{\alpha} \), under the “initial” condition

\[
\mathcal{X}(\frac{|x_0|}{\alpha}) = x_0 + |x_0| \ln \left( \frac{\rho_{jam}}{\rho_l} \right) = |x_0| \left[ -1 + \ln \left( \frac{\rho_{jam}}{\rho_l} \right) \right]
\]

To find the general integral of equation for the trajectory inside the fan the same procedure used for the linear model is employed.

A particular solution of the complete equation is

\[
\mathcal{X}_p(t) = \alpha t \ln t
\]
while the general solution of the homogeneous equation is

\[ \mathcal{X}_h(t) = Ct \]

The constant \( C \) is determined from the general solution of the nonhomogeneous complete equation

\[ \mathcal{X}(t) = \mathcal{X}_p(t) + \mathcal{X}_h(t) = Ct + \alpha t \ln t \]

The imposition of the “initial” condition allow to determine the constant \( C \)

\[ C = \alpha \left[ \ln \left( \frac{\rho_{\text{jam}} \alpha t}{\rho_l |x_0|} \right) - 1 \right] \]

from which the trajectory inside the fan is found to be

\[ \mathcal{X}(t) = \left[ \ln \left( \frac{\rho_{\text{jam}} \alpha t}{\rho_l |x_0|} \right) - 1 \right] \alpha t \]

for \( t \geq |x_0|/\alpha \). The complete expression of the trajectories of cars with \( x_0 < 0 \) is found to be

\[
\mathcal{X}(t) = \begin{cases} 
  x_0 + \left[ \ln \left( \frac{\rho_{\text{jam}} \alpha t}{\rho_l} \right) \right] \alpha t & 0 \leq t \leq \frac{|x_0|}{\alpha} \\
  \left[ \ln \left( \frac{\rho_{\text{jam}} \alpha t}{\rho_l |x_0|} \right) - 1 \right] \alpha t & \frac{|x_0|}{\alpha} \leq t \leq e \cdot \frac{\rho_l |x_0|}{\rho_r \alpha} \\
  \left[ \ln \left( \frac{\rho_{\text{jam}} \alpha t}{\rho_r} \right) \right] \alpha t & t \geq e \cdot \frac{\rho_l |x_0|}{\rho_r \alpha} 
\end{cases}
\]